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Normal Families in Complex Analysis

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This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

Abstract

The notion of a normal family of complex-valued functions was initially introduced by Paul Montel in 1907. Over the past century, normal families have become a pivotal concept in the field of complex analysis and are still an active area of research today. In this report we will study normal families of continuous and holomorphic functions, using our results to present a proof of the Riemann mapping theorem. We then generalise to normal families of meromorphic functions, studying various powerful conditions for normality. Finally, as a consequence of a particular normality criterion, we present a proof of the big Picard theorem: If a holomorphic function $f(z)$ has an isolated essential singularity at z_0, then on any punctured neighbourhood around z_0, $f(z)$ attains every value in $\mathbb{C}$ infinitely often, with possibly one exception.

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Preliminaries

Notation

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{C}_\infty$	the extended complex numbers $\mathbb{C} \cup \{\infty\}$
$\mathbb{D}$	the open unit disc $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
Ω	a domain (connected open set) contained in $\mathbb{C}$
$\Re(z)$	the real part x of a complex number $z = x + iy$
$\Im(z)$	the imaginary part y of a complex number $z = x + iy$
$\subset$	a (not necessarily proper) subset
$\overline{A}$	the closure of the set A
$\mathrm{Int}(A)$	the interior of the set A, the largest open subset of A
$B(a, r)$	an open ball $\{z \in \mathbb{C} : \lvert z - a\rvert < r\}$ in euclidean metric
$\overline{B}(a, r)$	a closed ball $\{z \in \mathbb{C} : \lvert z - a\rvert \leq r\}$ in euclidean metric
$B_d(y, r)$	an open ball in a metric space (Y, d), $\{y \in Y : d(y, y_1) < r, \quad y_1 \in Y\}$
$\overline{B}_d(y, r)$	a closed ball in the metric space (Y, d), $\{y \in Y : d(y, y_1) \leq r, \quad y_1 \in Y\}$
$\cong$	homeomorphism between two topological spaces
$\equiv$	identically equal to, equal to everywhere on the domain
$P\vert_D$	the map P with its domain restricted to D

All other relevant notation will be defined as it appears in the text.

Prerequisites

Readers should be familiar with notions of compactness, sequential compactness, relative compactness and completeness in metric spaces and topological spaces. Readers should be familiar with notions of continuity and convergence of sequences of points and functions. Of particular importance is compact convergence of a sequence of functions, and the fact that this preserves continuity. Readers should be familiar with standard results seen in a first course in complex analysis, including the construction of the Riemann sphere by stereographic projection. Explanations of these concepts and results can be found in textbooks such as [1, 7, 32].

1 Introduction

In the late 19th and early 20th century, the study of what would become known as compactness was in full force. One of the key properties under investigation was requiring every sequence $x_n \in X$ to have a convergent subsequence $x_{n_j} \to x$ - this is the Bolzano-Weierstrass property. If we ask further that the limit point x is in X, this is what modern analysis refers to as *sequential compactness*. Heuristically, if a set is sequentially compact then it is somehow small enough that it is not possible to have an infinite sequence inside it which does not have some accumulation point somewhere.

The Bolzano-Weierstrass theorem tells us that for subsets $X \subset \mathbb{R}^n$ or $\mathbb{C}^n$, X is sequentially compact if and only if it is both closed and bounded. A sketch proof of this on the interval $[0, 1] \subset \mathbb{R}$ is as follows. If we have an infinite sequence of points in $[0, 1]$, bisecting the interval there must be infinitely many terms in either $[0, 1/2]$ or $[1/2, 1]$. Choosing this and bisecting again, we have that at least one of the two intervals must contain infinitely many terms. We can repeat iteratively, with the length of the intervals 2^{-k} after the kth iteration, approaching zero. Taking one point in X from each interval gives a convergent subsequence.

It is easy to see how this argument generalises to $\mathbb{R}^n$. For an infinite closed and bounded sequence in $\mathbb{R}^n$, by boundedness find some hypercube which contains X completely. Partitioning this set into 2^n equal sub-hypercubes we can perform a very similar process to extract a subsequence. That sequential compactness implies closure and boundedness is easy to see. If we did not have closure then we would expose ourselves to the possibility that the limit function of this iteration was not in X. Furthermore, if we did not have boundedness then our bisection method would fall apart since the length is unbounded. For example, the sequence $x_n = n$ in $\mathbb{R}$ clearly has no convergent subsequence.

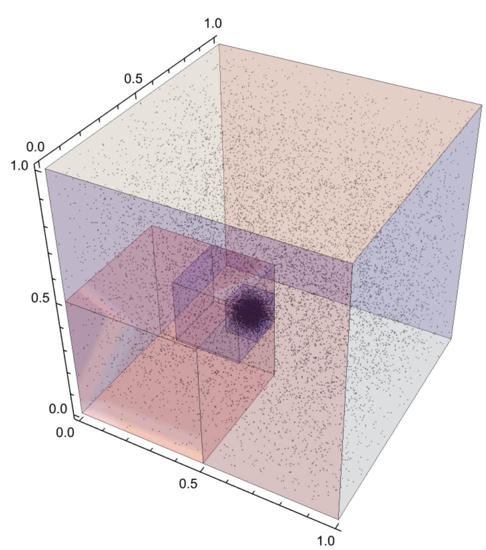

Figure 1.1: Bolzano-Weierstrass for an infinite set of points inside the unit cube in $\mathbb{R}^3$.

In 1904, French mathematician René Maurice Fréchet coined the term *compactness* for what is taken to be the modern definition - that X is compact if for every open covering of X, we can choose a finite collection of these to still cover X [10]. Fréchet made many major contributions to topology, including being the first to define the notion of a metric space. The Heine-Borel theorem tells us that a subset $X \subset \mathbb{R}^n$ or $\mathbb{C}^n$ is compact if and only if it is closed and bounded. We see that this aligns with the Bolzano-Weierstrass theorem, telling us compactness and sequential compactness are equivalent here. Although this is not true for general topological spaces, in a metric space, these two notions are equivalent. These classifications are powerful, but do not extend to infinite dimensional spaces. This is illustrated in the below example.

Example 1.1. Consider the space $\ell^p(\mathbb{R})$ for $1 < p < \infty$, the space of real-valued sequences, with the usual ℓ^p-norm. Let B be the closed unit ball

$$B = \{x : \|x\|_p \leq 1\} \subset \ell^p(\mathbb{R}).$$

Clearly, B is closed, as any convergent sequence $x_n \in B$ would also have its norm converging with each value $\|x_n\|_p \leq 1$ so the limit value must also satisfy this. By definition, it is also bounded. But the sequence $\{x_i\} \subset B$ given by

$$\begin{aligned} x_1 &= (1, 0, 0, 0, ...) \\ x_2 &= (0, 1, 0, 0, ...) \\ x_3 &= (0, 0, 1, 0, ...) \\ x_n &= (0, ..., 0, 1, 0, ...) \end{aligned}$$

has no convergent subsequence. To see this, observe that for $n \neq m$,

$$\|x_n - x_m\|_p = (|0-0|^p + ... + |1-0|^p + ...|0-1|^p + ...)^{1/p} = 2^{1/p} \geq 1 > 0.$$

No matter how large m and n are, we will never find a subsequence which converges, as the pairwise norms will never approach zero so the sequence is not Cauchy. Thus $B \subset \ell^p(\mathbb{R})$ is closed and bounded, but not sequentially compact.

So what went wrong? The problem was, in essence, that the space $\ell^p(\mathbb{R})$ was "too big". The infinite-dimensional size meant that in this case, closed and bounded subsets were too large that we could not force an accumulation point in every sequence.

This poses a serious problem in understanding when subsets of spaces of functions have this "closeness" Bolzano-Weierstrass property, as usually, function spaces are infinite dimensional, so any subset of them is likely to be too. Clearly being closed and bounded are necessary conditions for compactness, but what more must we ask in order to have an equivalent condition?

In 1895 this question was rigorously answered for subsets of the space of continuous functions, by the Italian mathematician Cesare Arzelà [3]. His work built on that of Giulio Ascoli's weaker proof of a sufficient condition for compactness of a collection of functions in 1883-84 [4]. We will see this in Chapter 2.

When we consider convergence properties, it is useful to begin thinking of functions as points in a space, rather than as mappings. To distinguish between these cases, the convention is to use the term *family*, when referencing a set of functions. Beyond this convention, there is no inherent difference in structure between a family and a set. We will usually consider a family of functions sharing a common domain $\Omega \subset \mathbb{C}$.

In the most general case in this report, we are interested in families of continuous functions and want to know when they are compact. In fact, we disregard the necessity of closure, as it is usually easy to see if this is the case. We are more concerned with if a convergent subsequence always exists, rather than if the limit of this subsequence is inside the family - this is the core property that tells us if the functions are "close together". In this way, we are more interested in studying families of functions which have the Bolzano-Weierstrass property. These are relatively compact families of functions - families with compact closure.

Implicit in all of this discussion is the mode of convergence of the functions, or equivalently, the choice of metric/topology on the space of functions. Restricting ourselves to uniform convergence limits the usefulness of results as they only apply to a small class of functions. Asking only pointwise convergence allows us to say little about the structure of any limit functions since it does not preserve continuity. The mode of convergence we will use is compact convergence - uniform convergence on all compact subsets of the domain. The topology this convergence induces is called the *compact-open* topology. Compact convergence is equivalent to local uniform continuity for all domains $\Omega \subset \mathbb{C}$, as can be seen by a simple Heine-Borel open covering compactness argument. It follows that compact convergence preserves continuity, which is a very useful property. We are now ready to define the eponymous notion of this report, a normal family of functions.

Definition 1.2. A family of continuous functions $\mathcal{F}$ with common domain Ω is a *normal family* if every sequence of functions $\{f_n\} \subset \mathcal{F}$ has a subsequence f_{n_k} which converges uniformly on every compact subset $X \subset \Omega$ to some $f_{n_k} \to f$. We *do not* require that the limit function of this subsequence satisfies membership $f \in \mathcal{F}$.

The functions in a normal family are clustered together enough that we have this Bolzano-Weierstrass property of always having an accumulation function in any sequence in $\mathcal{F}$. The study of normal families of complex-valued functions began in 1907, with Paul Montel giving an equivalent condition for normal families of holomorphic functions [21]. The term "normal family" was first used by Montel in a subsequent paper in 1911 [22].

Knowing when a family of functions is normal turns out to be very useful, as we will see in two main applications. In Chapter 4 we use the fact that a certain family is normal to aid us in proving a powerful result about the existence of biholomorphic mappings between certain domains - this is the Riemann mapping theorem. After developing our theory further, in Chapter 7 we see a further application of the theory of normal families, aiding us in proving one of the most celebrated results in complex analysis - the big Picard theorem.

2 Continuous Normal Families

In this first chapter, we discuss fundamental concepts regarding families of continuous functions. These concepts lay the bedrock for future chapters. We will be following and elucidating in more depth results predominantly from [1, 7, 29, 32], although this subject matter is seen in many other textbooks.

2.1 The space of continuous functions

We begin by studying families of continuous functions from subsets of the complex numbers to a general metric space. In this report, $(\Omega, |\cdot|_{\mathbb{R}^2})$ will always be used to represent the domain of our functions, where Ω is an open subset of $\mathbb{C}$ or $\mathbb{R}$, and $|\cdot|_{\mathbb{R}^2}$ is the Euclidean metric. In this chapter, the complete metric space (Y, d) will represent the codomain of our functions. We define $\mathcal{C}(\Omega, Y)$ as the set of all continuous functions from Ω to Y:

$$\mathcal{C}(\Omega, Y) := \{f : (\Omega, |\cdot|_{\mathbb{R}^2}) \to (Y, d) \mid f \text{ continuous}\}.$$

In the introduction, we defined the notion of a normal family, thinking of it as a relatively compact subset of $\mathcal{C}(\Omega, Y)$, with respect to the topology induced by compact convergence, the compact-open topology. However, we did not establish that this topology aligns with some metric on $\mathcal{C}(\Omega, Y)$. Metric spaces have lots of useful properties so we would like this to be the case. Crucially, what we are looking for is *not* the same as the difference between the values functions take, this is already given by $d(y_1, y_2)$, the metric on Y. We are after a more general notion of the distance between two functions in $\mathcal{C}(\Omega, Y)$ for which convergence in this metric is equivalent to uniform convergence on compact subsets $X \subset \Omega$.

It is not trivially the case that such a metric would exist, for example, there is no metric that aligns with the topology of pointwise convergence [9]. To prove this is challenging but it is easy to see that this would not give us a *complete* metric space since we know that pointwise convergence doesn't necessarily preserve continuity.

Our first step towards a metric inducing the compact-open topology is to reduce the infinitude of compact sets on which we must have convergence down to a countable collection.

Proposition 2.1. *[7] Let $\Omega \subset \mathbb{C}$ be open. There exists a sequence of compact subsets $X_k \subset \Omega$ with $k \in \mathbb{N}$, such that $\Omega = \cup_{k\geq 1} X_k$. We can further choose the sets such that*

(a) X_k are increasing, such that $X_k \subset \operatorname{Int}(X_{k+1})$ for all $k \in \mathbb{N}$.
(b) Any compact set $X \subset \Omega$ is contained in X_k for some $k \in \mathbb{N}$.

Such a sequence is called an exhaustion of Ω.

Proof. We know by Heine-Borel that any closed and bounded subset of $\mathbb{C}$ is compact, so we construct an increasing sequence of closed and bounded sets with limit Ω. Define X_k by

$$X_k := \{z : |z| \leq k\} \cap \{z : |z - w| \geq 1/k \text{ for all } w \in \mathbb{C} \setminus \Omega\}.$$

Each X_k is bounded by k and closed, as it is the intersection of two closed sets. Furthermore, $z \in X_k$ implies

$$|z| \leq k < k + 1 \text{ and } |z - w| \geq \frac{1}{k} > \frac{1}{k+1} \text{ for all } w \in \mathbb{C} \setminus \Omega,$$

so the sequence is increasing. If $z \in \Omega$ then for suitably large k we have $|z| \leq k$. By openness, we can find some $B(z, \delta) \subset \Omega$. Then for all $w \in \mathbb{C} \setminus \Omega$, $|z - w| \geq \delta$ and so for suitably large k we have $|z - w| \geq 1/k$ for all $w \in \mathbb{C} \setminus \Omega$. Thus, for suitably large k we have $z \in X_k$, giving $\Omega \subset \cup_{k\geq 1} X_k$. If $z \in \mathbb{C} \setminus \Omega$ then with $w = z$ we have $|w - z| = 0$ for some $w \in \mathbb{C} \setminus \Omega$ and so z is contained in no X_k and thus not in the limit. It follows that $\Omega = \cup_{k\geq 1} X_k$.

For arbitrary $X \subset \Omega$ compact, the open sets $\{\text{Int}(X_k)\}_{k=1}^{\infty}$ form an open cover of X and thus we have a finite subcover $\{\text{Int}(X_{k_i})\}_{i=1}^{n}$. Then $X \subset \text{Int}(X_{k_n})$ due to the increasing nature of the sets, so X is contained in some X_k as required. □

Equipped with such an exhaustion, we now construct a metric for the space of functions $\mathcal{C}(\Omega, Y)$. Let X_k be an exhaustion of Ω. Observe that $f_n \to f$ uniformly on compact subsets on Ω if and only if $f_n \to f$ uniformly on all X_k, since every $X \subset \Omega$ compact is in some X_k.

Requiring $f_n \to f$ uniformly on X_k is the same as asking that the supremum value of $d(f_n(z), f(z))$ over the entire $z \in X_k$ can be made arbitrarily small, such that the quantity

$$\sup_{z \in X_k} d(f_n(z), f(z)) \to 0.$$

Since we want a sequence in our metric to converge if and only if we have compact convergence, a sensible choice would be to consider a sum of all the above suprema. Then, if there was some compact $X \subset \Omega$ on which f_n did not converge uniformly to f, this X would be in some X_k and the suprema for this set would not go to 0, so neither would the sum. However, we could easily encounter a case where each expression converged to zero while the sum diverged. To force the sum to 0 in this case we make some modifications. Our first is to replace the metric d with a bounded metric, with help from [16].

Lemma 2.2. *Let (Y, d) be a metric space. Then the quantity $\delta(y_1, y_2)$ is a metric on Y, defined by*

$$\delta(y_1, y_2) := \frac{d(y_1, y_2)}{1 + d(y_1, y_2)}.$$

Furthermore, for any sequence $y_n \in Y$ we have that $\delta(y_n, y) \to 0$ if and only if $d(y_n, y) \to 0$.

Proof. Positivity and symmetry of the metric are trivially inherited from d. For the triangle inequality, we note that the function

$$f(t) = \frac{t}{1+t} \quad f'(t) = \frac{1}{(1+t)^2}$$

is continuous and has positive derivative for $t \geq 0$, so $f(t)$ is strictly increasing here. Thus

$$\begin{aligned}
\delta(y_1, y_2) = f(d(y_1, y_2)) \leq & f(d(y_1, y_3) + d(y_2, y_3)) \\
= & \frac{d(y_1, y_3)}{1 + d(y_1, y_3) + d(y_2, y_3)} + \frac{d(y_2, y_3)}{1 + d(y_1, y_3) + d(y_2, y_3)} \\
\leq & \frac{d(y_1, y_3)}{1 + d(y_1, y_3)} + \frac{d(y_2, y_3)}{1 + d(y_2, y_3)} \\
= & \delta(y_1, y_3) + \delta(y_2, y_3).
\end{aligned}$$

Where we made use of the triangle inequality for d in the first line. This establishes $\delta(y_1, y_2)$ as a metric on Y. The limit property is easy to see. We first assume $d(y_n, y) \to 0$, and note this is an upper bound for $\delta(y_n, y)$ since the denominator of $\delta(y_n, y)$ is greater than or equal to 1.

$$0 \leq \delta(y_n, y) \leq \frac{d(y_n, y)}{1 + d(y_n, y)} \leq d(y_n, y) \to 0$$

so $\delta(y_n, y) \to 0$ by squeezing. Now assume $\delta(y_n, y) \to 0$, thus there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\delta(y_n, y) < 1/2$. By rearranging the definition for δ, we obtain

$$0 \leq d(y_n, y) = \frac{\delta(y_n, y)}{1 - \delta(y_n, y)} \leq \frac{\delta(y_n, y)}{1 - 1/2} = 2\delta(y_n, y) \to 0.$$

Thus $\delta(y_n, y) \to 0$ if and only if $d(y_n, y) \to 0$ as required. □

Effectively what we have shown here is that these two metrics are *equivalent*, inducing the same topology on Y. We discuss this notion in more detail in Chapter 5. The advantage of the metric δ is that it is bounded: $0 \leq \delta(y_1, y_2) \leq 1$. Thus an infinite sum of these quantities is much easier to make converge. We define

$$\Delta_k(f, g) := \sup_{z \in X_k} \delta(f(z), g(z)) = \sup_{z \in X_k} \frac{d(f(z), g(z))}{1 + d(f(z), g(z))}.$$

Now we define our metric on $\mathcal{C}(\Omega, Y)$ as a sum of these, with a geometric factor 2^{-k} to force the convergence.

$$\rho(f, g) := \sum_{k=1}^{\infty} 2^{-k} \Delta_k(f, g); \quad f, g \in \mathcal{C}(\Omega, Y).$$

Theorem 2.3. *The set $\mathcal{C}(\Omega, Y)$ equipped with the metric $\rho(f, g)$ is a complete metric space, and convergence in this metric is equivalent to uniform convergence on all compact $X \subset \Omega$.*

Proof. Since $\rho(f, g)$ is a sum of suprema of metrics then clearly $\rho(f, g) \geq 0$. If $\rho(f, g) = 0$ then $f(z) = g(z)$ on every compact $X \subset \Omega$. Since every $z \in \Omega$ is in some compact set - for example, the singular $\{z\}$ - then $f(z) = g(z)$ on Ω as required. Symmetry of the metric is trivially inherited from d by construction. The triangle inequality is also inherited from d as follows. Let $f, g, h \in \mathcal{C}(\Omega, Y)$ and fix some k, then

$$\begin{aligned} \Delta_k(f, h) + \Delta_k(g, h) &= \sup_{z \in X_k} \delta(f(z), h(z)) + \sup_{z \in X_k} \delta(g(z), h(z)) \\ &\geq \sup_{z \in X_k} (\delta(f(z), h(z)) + \delta(g(z), h(z))) \\ &\geq \sup_{z \in X_k} \delta(f(z), g(z)) = \Delta_k(f, g). \end{aligned}$$

Since this is true for all $k \in \mathbb{N}$ then considering the full series we have the triangle inequality:

$$\rho(f, h) + \rho(g, h) = \sum_{k=1}^{\infty} 2^{-k} (\Delta_k(f, h) + \Delta_k(g, h)) \geq \sum_{k=1}^{\infty} 2^{-k} \Delta_k(f, g) = \rho(f, g).$$

This establishes ρ as a metric on $\mathcal{C}(\Omega, Y)$.

To confirm this metric aligns with uniform convergence on compact subsets, observe that $f_n \to f$ compactly on Ω if and only if each $\Delta_k(f_n, f) \to 0$ as $n \to \infty$ for each fixed $k \in \mathbb{N}$.

For each fixed n, the sum for $\rho(f_n, f)$ is dominated by the geometric series since the metric δ is bounded by 1:

$$\rho(f_n, f) = \sum_{k=1}^{\infty} 2^{-k} \Delta_k(f, g) \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

By the dominated convergence theorem it follows that

$$\lim_{n\to\infty} \rho(f_n, f) = \lim_{n\to\infty} \sum_{k=1}^{\infty} 2^{-k} \Delta_k(f_n, f) = \sum_{k=1}^{\infty} 2^{-k} \lim_{n\to\infty} \Delta_k(f_n, f)$$

and thus since $\Delta_k(f_n, f) \geq 0$ for all $n, k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \rho(f_n, f) \iff \Delta_k(f_n, f) \to 0 \text{ for all } n \in \mathbb{N} \text{ for each fixed } k.$$

This gives equivalence of convergence in ρ to compact convergence. For completeness, note that compact convergence preserves continuity, so any Cauchy sequence of continuous functions will have a limit function that is also continuous. □

Theorem 2.4. *A family $\mathcal{F} \subset \mathcal{C}(\Omega, Y)$ is a normal family if and only if its closure with respect to ρ is compact.*

Proof. We first show that $\mathcal{F}$ normal implies $\overline{\mathcal{F}}$ is normal. Assume $\mathcal{F}$ is normal and let $f_n \in \overline{\mathcal{F}}$ and fix $\epsilon > 0$. By nature of the closure[1], we can find functions $g_n \in \mathcal{F}$ such that $\rho(g_n, f_n) < 1/n$ for all $n \in \mathbb{N}$. Since $\mathcal{F}$ is normal we have a convergent subsequence $g_{n_k} \to g \in \overline{\mathcal{F}}$, so there exists $N \in \mathbb{N}$ such that $\rho(g_{n_k}, g) < \epsilon/2$ for all $n_k \geq N$. Now by the triangle inequality, for $n_k \geq \max\{N, 2/\epsilon\}$

$$\rho(f_{n_k}, g) \leq \rho(g_{n_k}, f_{n_k}) + \rho(g_{n_k}, g) < \epsilon/2 + \epsilon/2 = \epsilon,$$

and so $f_{n_k} \to g$. Thus every sequence in $\overline{\mathcal{F}}$ has a convergent subsequence so $\overline{\mathcal{F}}$ is normal. Since $\mathcal{F} \subset \overline{\mathcal{F}}$, the set of all sequences in $\mathcal{F}$ is contained in that of $\overline{\mathcal{F}}$, so $\overline{\mathcal{F}}$ normal clearly implies $\mathcal{F}$ is normal. So $\mathcal{F}$ is normal if and only if its closure $\overline{\mathcal{F}}$ is.

If $\overline{\mathcal{F}}$ is normal, every sequence has a convergent subsequence with the limit in $\overline{\mathcal{F}}$ by closure and completeness of $\mathcal{C}(\Omega, Y)$. But this is the definition for sequential compactness, so since we are in a metric space $\overline{\mathcal{F}}$ compact. Thus we are done, as we have shown

$$\mathcal{F} \text{ normal} \iff \overline{\mathcal{F}} \text{ normal} \iff \overline{\mathcal{F}} \text{ compact}.$$

□

This confirms clearly what the notion of a normal family intuitively is, a subset of the space of continuous functions which is relatively compact in the compact-open topology. As we saw in the above proof, it is trivially the case that any subfamily of a normal family will be normal, just as subsets of relatively compact sets are themselves relatively compact.

[1] The closure is $\mathcal{F}$ and its limit points so each f_n is in $\mathcal{F}$ or arbitrarily close to a function in $\mathcal{F}$.

2.2 Equicontinuous families of functions

In pursuit of an equivalence condition for compactness in the space of continuous functions, we must first define the notion of *equicontinuity.* This will be a crucial notion in the condition for normality presented in the next section. Equicontinuity is a stronger form of continuity across the entire family. In the same way as uniform continuity demands that for each $\epsilon > 0$, we can find a $\delta > 0$ which gives continuity at all points in the domain, a family is equicontinuous at a point in the domain if we can find a δ which gives continuity at that point for all functions in the family.

Definition 2.5. A family $\mathcal{F} \subset \mathcal{C}(\Omega, Y)$ is *equicontinuous* at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and $z \in \Omega$ with $|z - z_0| < \delta$, we have $d(f(z), f(z_0)) < \epsilon$. A family $\mathcal{F}$ is *equicontinuous over a set* $X \subset \Omega$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $z_1, z_2 \in X$ we have $d(f(z_1), f(z_2)) < \epsilon$.

The subtleness of the definition must be carefully observed here. Being equicontinuous over a set is *not* the same as being equicontinuous at each point in the set. It is instead a notion of being uniformly continuous with the same δ working for every function also. Equicontinuity is illustrated in the following examples.

Example 2.6. Consider the family of continuous functions $\mathcal{F} \subset \mathcal{C}(\mathbb{C}, \mathbb{C})$ given by

$$\mathcal{F} = \{f(z) = z^2 + n : n \in \mathbb{N}\}.$$

Fixing $\epsilon > 0$ we see that $\mathcal{F}$ is equicontinuous at each $z_0 \in \mathbb{C}$ by choosing

$$\delta = \min\left\{1, \frac{\epsilon}{1 + 2|z_0|}\right\}.$$

Then for $|z - z_0| < \delta$, observe that for all $f \in \mathcal{F}$, we have

$$\begin{aligned}
|f(z) - f(z_0)| &= |z^2 + n - (z_0^2 + n)| = |z + z_0||z - z_0| \\
&\leq (|z_0| + |z|)|z - z_0| \\
&< (|z_0| + |z_0| + \delta)|z - z_0| \\
&\leq (2|z_0| + 1)\delta \leq \epsilon,
\end{aligned}$$

giving equicontinuity at an arbitrary $z_0 \in \mathbb{C}$. But $\mathcal{F}$ is not equicontinuous over $\mathbb{C}$. Fix $\epsilon = 1$ and assume there exists such a $\delta > 0$ which works for all $f \in \mathcal{F}$ and all $z \in \mathbb{C}$. The point $z + \delta/2$ is within δ of z so for all $z \in \mathbb{C}$ we should have

$$|f(z) - f(z + \delta/2)| = |z^2 + n - (z + \delta/2)^2 - n| = |\delta^2/4 + z\delta| < 1.$$

Choosing z suitably large would clearly make this modulus greater than 1 so this is a contradiction, and it follows that no such δ exists. Here $\mathcal{F}$ failed to be equicontinuous over $\mathbb{C}$ since each function was not uniformly continuous over $\mathbb{C}$. Noting that continuous functions are uniformly continuous on compact sets, we can establish $\mathcal{F}$ is equicontinuous over any compact $X \subset \mathbb{C}$ however. Let $X \subset \mathbb{C}$ be compact, then X is bounded, say by $|z| \leq M$. We know that $\mathcal{F}$ is equicontinuous at each point in $z \in X$ with the δ above. Thus taking

$$\delta = \min\left\{1, \frac{\epsilon}{1 + 2M}\right\}$$

gives $\mathcal{F}$ equicontinuous at any point $z \in X$, so $\mathcal{F}$ is equicontinuous over every compact $X \subset \mathbb{C}$.

Example 2.7. Consider the family of continuous functions $\mathcal{F} \subset \mathcal{C}((0,1), \mathbb{R})$ given by

$$\mathcal{F} = \{f(x) = nx : n \in \mathbb{N}\}.$$

By choosing $\delta = \epsilon/n$ it is clear each function in $\mathcal{F}$ is uniformly continuous on $(0,1)$. However, $\mathcal{F}$ is not equicontinuous at any point in $(0,1)$. Fix $\epsilon = 1$ and assume without loss of generality that there is a $\delta < 1/2$ which works for some $x_0 \in (0,1)$. Then for all $f \in \mathcal{F}$,

$$|f(x) - f(x_0)| = n|x - x_0| < 1 \quad \text{whenever } |x - x_0| < \delta.$$

But for $n > 2/\delta$ this fails for $x = x_0 \pm \delta/2$, whichever or both are in $(0,1)$. So in this case it is not the uniformity across the interval that fails, but the equicontinuity altogether, the uniformity across the functions.

We can see that equicontinuity is a property of closeness of a family, since requiring that all functions can use the same ϵ for continuity requires that at each point, every function is of similar shape/derivative. Then, a small change δ in the inputs of the functions corresponds to a change in the outputs which is bounded across all functions. In essence, this is why Example 2.7 failed, as the functions could become infinitely steep on $(0,1)$ as n is made large.

In Example 2.6, we saw that $\mathcal{F}$ was pointwise equicontinuous, and on a compact set X, we could choose a minimum of all the deltas at each point $z_0 \in X$ such that $\mathcal{F}$ was in fact equicontinuous over X. This was not a coincidence and is true in the general case.

Proposition 2.8. *If a family $\mathcal{F} \subset \mathcal{C}(\Omega, Y)$ is equicontinuous at each point $z \in X$ for a compact set $X \subset \Omega$, then $\mathcal{F}$ is equicontinuous over X.*

Proof. Fix $\epsilon > 0$. For each $z \in X$, $\mathcal{F}$ is equicontinuous so there exists $\delta_z > 0$ for which $|f(z) - f(w)| < \epsilon$ whenever $|z - w| < \delta_z$ and $w \in X$ and $f \in \mathcal{F}$. The set

$$\mathcal{O} = \{B(z, \delta_z) \mid z \in X\}$$

forms an open covering of X, since each z is at least in its own ball. By compactness, there exists a finite subcovering

$$X \subset \bigcup_{i=1}^{n} B(z_i, \delta_{z_i})$$

for some finite sets $\{z_i\} \subset X$ and $\{\delta_{z_i}\}$. We now set

$$\delta = \min_{1 \leq i \leq n} \{\delta_{z_i}\}.$$

Then any $z \in X$ is contained in some $B(z_i, \delta_{z_i})$, so if $|z - z_i| < \delta \leq \delta_{z_i}$ then by equicontinuity at z_i, $|f(z) - f(z_i)| < \epsilon$ for all $f \in \mathcal{F}$. Since this δ works for all $z \in X$, then $\mathcal{F}$ is equicontinuous over X as required. □

2.3 The Arzelà–Ascoli theorem

We now prove the powerful Arzelà–Ascoli theorem, which makes use of equicontinuity to give us our first condition for normality, and is a fundamental building block towards the proof of many of our later results. Here, we follow proof from [1, 7, 34].

Theorem 2.9 (The Arzelà–Ascoli Theorem)**.** *A family $\mathcal{F} \subset \mathcal{C}(\Omega, Y)$ is normal if and only if both of the following are true:*

(i) $\mathcal{F}$ is equicontinuous over every compact $X \subset \Omega$.
(ii) The sets $\{f(z_0) : f \in \mathcal{F}\}$ are relatively compact in Y for each $z_0 \in \Omega$.

Proof. Assume $\mathcal{F}$ is normal. We will first show $\mathcal{F}$ is equicontinuous at each $z_0 \in \Omega$. We argue by contradiction, assuming $\mathcal{F}$ is not equicontinuous at z_0. If for each $\epsilon > 0$, there were a δ which worked for equicontinuity at z_0 for all but finitely many functions $\{f_i\} \subset \mathcal{F}$, then by continuity of each function we could find a δ_i giving continuity at z_0. Taking a minimum of all the δ_i and the original δ would give us a δ which would suffice for equicontinuity at z_0. So, since we are assuming no equicontinuity, there must exist some $\epsilon > 0$ such that for all $\delta > 0$, there is an infinite sequence of functions $\{f_n\} \subset \mathcal{F}$ such that for all $n \in \mathbb{N}$,

$$d(f_n(z_0), f_n(z)) \geq \epsilon; \qquad \text{for some } z \in \Omega \text{ with } |z_0 - z| < \delta. \tag{2.1}$$

We fix this ϵ from above, and now find by normality a subsequence of this sequence which converges, using this to force some $d(f_n(z_0), f_n(z))$ to be less than ϵ. This in turn contradicts our choice that no f_n satisfies this, that lack of equicontinuity allowed us to make. By normality, we have a convergent subsequence $f_{n_k} \to f \in \mathcal{C}(\Omega, Y)$. By the triangle inequality,

$$d(f_{n_k}(z_0), f_{n_k}(z)) \leq d(f_{n_k}(z_0), f(z_0)) + d(f(z_0), f(z)) + d(f(z), f_{n_k}(z)).$$

By continuity of f and twice by the convergence of f_{n_k} there exists $\delta > 0$ (simply by taking the minimum of the two deltas required for each) and $N \in \mathbb{N}$ such that for $|z - z_0| < \delta$ and $n_k \geq N$, we have

$$d(f_{n_k}(z_0), f(z_0)) < \epsilon/3, \quad d(f(z_0), f(z)) < \epsilon/3, \quad d(f(z), f_{n_k}(z)) < \epsilon/3.$$

Thus for $|z - z_0| < \delta$ and $n_k \geq N$ we have

$$d(f_{n_k}(z_0), f_{n_k}(z)) < \epsilon.$$

This is a contradiction to (2.1). Therefore $\mathcal{F}$ is equicontinuous at each $z_0 \in \Omega$. By Proposition 2.8, $\mathcal{F}$ is equicontinuous over every compact $X \subset \Omega$, establishing condition (i).

To show condition (ii), we show that $W = \{f(z_0) : f \in \mathcal{F}\}$ has compact closure in Y, using a similar argument to the proof of Theorem 2.4. Fix $\epsilon > 0$ and let $y_n \in \overline{W}$ be a sequence in the closure, then there is a sequence $f_n(z_0) \in W$ for some sequence $f_n \in \mathcal{F}$, such that $d(f_n(z_0), y_n) < 1/n$. By normality, there is a subsequence $f_{n_k} \to f$ which converges uniformly on the compact set $\{z_0\} \subset \Omega$. Thus there exists $N \in \mathbb{N}$ such that for all $n_k \geq \max\{2/\epsilon, N\}$,

$$d(y_{n_k}, f(z_0)) \leq d(y_{n_k}, f_{n_k}(z_0)) + d(f_{n_k}(z_0), f(z_0)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

So $y_{n_k} \to f(z_0) \in \overline{W}$, meaning $\overline{W}$ is sequentially compact and thus compact as we are in a metric space. Thus W is relatively compact. This establishes one direction of the proof.

We now assume conditions (i) and (ii), and deduce normality. There exists a countable dense subset of Ω, for example, all the points with rational real and imaginary parts. Let $\{z_m\}_{m \in \mathbb{N}} \subset \Omega$ be an enumeration of this set, so for every point $z \in \Omega$, we can find a z_m arbitrarily close to z. If for each sequence $f_n \in \mathcal{F}$ we can find a subsequence f_{n_k} which converges at all $z_m \in \Omega$, then by density this sequence must converge at every $z \in \Omega$.

Let $\{f_n\} \subset \mathcal{F}$. By condition (ii) the sequence $\{f_n(z_1)\}_{n=1}^{\infty}$ has a convergent subsequence, we will write this as $\{f_{1,n}(z_1)\}_{n=1}^{\infty}$, such that the sequence of functions $f_{1,n}$ converge pointwise to some value at z_1. By condition (ii) again, we now take another subsequence $\{f_{2,n}(z_1)\}_{n=1}^{\infty}$ of these this previous sequence, which now converges at z_1 and z_2. We can repeat this process iteratively to construct a subsequence which converges at all $\{z_1, z_2, .., z_m\}$. we express each sequence below where each row is a subsequence of the row above.

$$\begin{matrix} f_{1,1} & f_{1,2} & f_{1,3} \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} \cdots \\ \vdots & \vdots & \vdots \ddots \end{matrix}$$

The first-row sequence converges at z_1, the second row converges at z_1 and z_2, the third at z_1, z_2 and z_3 and so on. By taking the subsequence along the diagonals, $f_{n_k} := f_{n,n}$ we have a subsequence of the original sequence which converges at every point $z_m \in \Omega$. Since these points are dense and each f_{n_k} is continuous, it follows that f_{n_k} converges pointwise everywhere in Ω. For clarity, we will now refer to this convergent subsequence as f_k.

For the convergence to be uniform on compacta, we make use of equicontinuity from condition (i). Since pointwise convergence tells us little about the limit function's behaviour, we will show that the sequence f_k is uniformly Cauchy and thus uniformly convergent, since $\mathcal{C}(\Omega, Y)$ is complete. Let $X \subset \Omega$ be compact and fix $\epsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that for all $z, z_0 \in X$ and $f \in \mathcal{F}$, $|z - z_0| < \delta$ implies

$$d(f(z), f(z_0)) < \epsilon/3. \tag{2.2}$$

By compactness, there exists a finite subcovering of X with balls of size $\delta/2$

$$X \subset \bigcup_{r=1}^{l} B(\zeta_r, \delta/2); \quad \zeta_r \in X \text{ and } l \in \mathbb{N}.$$

By pointwise convergence (and thus pointwise Cauchy), for each ζ_r there exists an $M_r > 0$ for which $d(f_{k_i}(\zeta_r), f_{k_j}(\zeta_r)) < \epsilon/3$ for $k_i, k_j \geq M_r$.[2] Since there are only finitely many ζ_r, taking $M = \max_{1 \leq r \leq l} M_r$ gives us that for all $k_i, k_j \geq M$ we have

$$d(f_{k_i}(\zeta_r), f_{k_j}(\zeta_r)) < \epsilon/3.$$

Since each $z \in X$ is in one of the finite $\delta/2$ balls of our covering, each z has a ζ_r for which $|z - \zeta_r| < \delta$ and thus by (2.2)

$$d(f_{k_i}(z), f_{k_i}(\zeta_r)) < \epsilon/3 \quad \text{and equivalently} \quad d(f_{k_j}(z), f_{k_j}(\zeta_r)) < \epsilon/3.$$

Thus for each $z \in X$ and its corresponding ζ_r, and for $k_i, k_j \geq M$

$$\begin{aligned} d(f_{k_i}(z), f_{k_j}(z)) \leq & d(f_{k_i}(z), f_{k_i}(\zeta_r)) + d(f_{k_i}(\zeta_r), f_{k_j}(\zeta_r)) + d(f_{k_j}(z), f_{k_j}(\zeta_r)) \\ < & \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

This concludes the proof, as since M is independent of z then f_k is uniformly Cauchy, and thus uniformly convergent on X since $\mathcal{C}(\Omega, Y)$ is complete. Thus every sequence in $\mathcal{F}$ has a subsequence that converges compactly, so $\mathcal{F}$ is normal as required. □

[2] Here, k_i and k_j represent two different terms in the sequence f_k, used for the Cauchy property.

Both of the conditions in Arzelà–Ascoli are local conditions, since by Proposition 2.8 condition (i) is equivalent to being equicontinuous at each point. Thus, it is apparent that the normality of a family is a local property. We consider a family $\mathcal{F}$ *normal at a point* $z_0 \in \Omega$ if it is normal on some neighbourhood of z_0. Then, $\mathcal{F}$ fails to be normal on Ω if there is some point in Ω at which $\mathcal{F}$ is not normal.

Corollary 2.10. *A family $\mathcal{F} \subset \mathcal{C}(\Omega, Y)$ is normal if and only if it is normal at each $z \in \Omega$.*

Proof. It is clear that if we have equicontinuity on a neighbourhood of each point then we have pointwise and thus equicontinuity over compacta of Ω, for Arzelà–Ascoli condition (i). The converse is trivially true. Arzelà–Ascoli condition (ii) on a neighbourhood of each point is trivially equivalent to being satisfied on all Ω. □

Example 2.11. Consider the family of continuous functions $\mathcal{F} \subset \mathcal{C}((-1,1), \mathbb{R})$ given by

$$\mathcal{F} = \{f(x) = \sin(nx) : n \in \mathbb{N}\}.$$

Since for each $x \in \mathbb{R}$ the sets $\{\sin(nx) \mid n \in \mathbb{N}\}$ are contained in $[-1,1]$, they are bounded and thus relatively compact so condition Arzelà–Ascoli condition (ii) holds. However, $\mathcal{F}$ fails to be equicontinuous on compacta - take the interval $[0,1/2] \subset (-1,1)$ and fix $\epsilon = 1/2$. Arguing by contradiction we assume there is an $\delta > 0$ such that for all $n \in \mathbb{N}$, $|\sin(nx_1) - \sin(nx_2)| < 1/2$ for all $|x_1 - x_2| < \delta$ inside $[0,1/2]$. If such a δ exists, then any smaller δ also work, so we can assume $\delta < 1/2$. Choose $N \in \mathbb{N}$ such that $\pi/(2N) < \delta$, then taking $x_1 = 0$ and $x_2 = \pi/(2N)$ gives

$$|\sin(N0) - \sin(N\pi/(2N))| = |\sin(\pi/2)| = 1 < \frac{1}{2}.$$

This is a clear contradiction, so $\mathcal{F}$ is not equicontinuous here. A similar argument shows equicontinuity fails everywhere on the real line for this family, as we can find suitably large n such that $\sin(nx)$ is arbitrarily steep on any neighbourhood of any point, contradicting any δ which would potentially work there. It follows that $\mathcal{F}$ is not normal on $(-1,1)$, or in fact more generally anywhere in $\mathbb{R}$.

Example 2.12. Consider the family of continuous functions $\mathcal{F} \subset \mathcal{C}(\mathbb{D}, \mathbb{C})$ given by

$$\mathcal{F} = \left\{ f(z) = \frac{1}{z+n} : n \in \mathbb{N} \right\}.$$

It is clear that for each n the supremum value of each function on $\mathbb{D}$ is approached as we approach $z = -1$ on the boundary of $\mathbb{D}$. In the case $n = 1$ this limit diverges to infinity. The points in any compact set $X \subset \mathbb{D}$ are bounded away from -1, so the functions are defined on all of $\mathbb{D}$. Let $X \subset \mathbb{D}$ be compact and let $r > 0$ be the minimum Euclidean distance between -1 and any point in X. For all $n \in \mathbb{N}$ (and consequently all $f \in \mathcal{F}$) and $z, w \in X$

$$|f(z) - f(w)| = \left| \frac{1}{z+n} - \frac{1}{w+n} \right| = \left| \frac{w-z}{(z+n)(w+n)} \right| \leq \frac{1}{r^2}|z-w|.$$

Thus for fixed $\epsilon > 0$, picking $\delta = r^2\epsilon/2$ gives equicontinuity over X, Arzelà–Ascoli condition (i). Each $z \in \mathbb{D}$ is contained in some X compact and thus the functions are bounded at z by $1/r$ above, so Arzelà–Ascoli condition (ii) holds also. Thus by Arzelà–Ascoli $\mathcal{F}$ is normal on $\mathbb{D}$.

3 Holomorphic Normal Families

In the previous chapter, we developed an understanding of the space of continuous functions $\mathcal{C}(\Omega, Y)$, and established conditions required for subsets of this space to be relatively compact in the compact-open topology, as a normal family. This chapter begins by studying the smaller subspace of holomorphic functions. We then give an equivalent condition for the normality of a family in this space.

3.1 The space of holomorphic functions

We define the space of holomorphic functions on Ω by

$$\mathcal{H}(\Omega) := \{f : (\Omega, |\cdot|_{\mathbb{R}^2}) \to (\mathbb{C}, |\cdot|_{\mathbb{R}^2}) \mid f \text{ holomorphic}\}.$$

Since holomorphic functions are continuous, it is clear that $\mathcal{H}(\Omega) \subset \mathcal{C}(\Omega, \mathbb{C})$. This means we can make use of the same metric $\rho(f, g)$ we established in Chapter 2. Recall that our proof of Arzelà–Ascoli established that the two conditions implied every sequence must have a uniformly Cauchy subsequence, and we used the completeness of $\mathcal{C}(\Omega, Y)$ to deduce the existence of a limit function. Thus, we need $\mathcal{H}(\Omega)$ to be complete if we want to make use of Arzelà–Ascoli on this subspace. This is indeed the case, as we now prove, following [1, 7].

Theorem 3.1 (Weierstrass's Theorem)**.** *Let $\{f_n\} \subset \mathcal{H}(\Omega)$ be a sequence of holomorphic functions which converge compactly to some $f_n \to f$ on Ω. Then f is holomorphic on Ω, that is $f \in \mathcal{H}(\Omega)$.*

Proof. Let $z_0 \in \Omega$ be arbitrary, then by openness of Ω, we can find a closed ball $\overline{B}(z_0, r) \subset \Omega$. This ball is compact so we have uniform convergence here. If γ is a closed curve in $\overline{B}(z_0, r)$ then by Cauchy's theorem, for all $n \in \mathbb{N}$ we have

$$\int_\gamma f_n(z)dz = 0.$$

Taking the limit as $n \to \infty$ by uniform convergence we deduce

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \int_\gamma f(z)dz = 0$$

for any closed contour $\gamma \subset \overline{B}(z_0, r)$. Thus by Morera's theorem, f is holomorphic at z_0, so f is holomorphic on Ω since z_0 was arbitrary. □

Corollary 3.2. *$\mathcal{H}(\Omega)$ equipped with the metric $\rho(f, g)$ for $\mathcal{C}(\Omega, \mathbb{C})$ is a complete metric space.*

We now know we can apply Arzelà–Ascoli on $\mathcal{H}(\Omega)$. Weierstrass's theorem shows us the power of compact convergence, preserving not only continuity but holomorphicity also. In fact, more structure is preserved, as we see in the below theorem due to A. Hurwitz.

Theorem 3.3 (Hurwitz's Theorem)**.** *Let $\{f_n\} \subset \mathcal{H}(\Omega)$ be a sequence of compactly convergent holomorphic functions with limit $f_n \to f$, and let $a \in \mathbb{C}$. If $f_n(z) \neq a$ on Ω for all $n \in \mathbb{N}$ then either f is constant with $f(z) \equiv a$ on Ω, or $f(z) \neq a$ on Ω.*

Proof. By considering the sequence $g_n(z) = f_n(z) - a$ with limit $g(z) = f(z) - a$, we can assume without loss of generality that $a = 0$. Suppose f is not identically zero. f is holomorphic since compact convergence preserves holomorphicity, and so the zeros of f must be isolated (this is a well-known property of holomorphic functions).

The openness of Ω and the principle of isolated zeros tell us that for a zero $z_0 \in \Omega$ of f, we can find $r > 0$ such that $f(z) \neq 0$ on the closed punctured disc $\{z : 0 < |z - z_0| \leq r\}$. More specifically, on the circle $|z - z_0| = r$, $f(z) \neq 0$. Since this circle is compact then f is bounded away from zero, so $1/f_n$ converges uniformly to $1/f$ here also. We also know that $f_n' \to f'$ uniformly on this compact set. Thus the product of $1/f$ and f' converges uniformly here also. It follows that

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_{|z-z_0|=r} \lim_{n\to\infty} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'(z)}{f(z)} dz.$$

The argument principle tells us that for each n the integral on the left is the number of zeros contained inside the circle, so must equal zero since $f_n(z) \neq 0$ on all Ω for all $n \in \mathbb{N}$. But the integral on the right tells us the number of zeros f has inside this circle so f must also be nonzero here, contradicting that z_0 was a zero of f. It follows that $f(z) \neq 0$ on Ω as required. □

Hurwitz's theorem will be crucial to many proofs throughout this report. That compact convergence of sequences of holomorphic functions must be so well-behaved is very useful. We see how the result does not necessarily hold in the real-valued case, and confirm the result in two cases with the examples below.

Example 3.4. Consider the real-valued continuous functions

$$f_n : \mathbb{R} \to \mathbb{R} \quad f_n(x) = x^2 + 1/n \to f(x) = x^2.$$

Each function is never zero on $\mathbb{R}$, but the limit function $f(x) = x^2$ has a zero. If we instead considered these functions with a holomorphic domain containing zero, then each function would have a zero $f(i/\sqrt{n}) = 0$, so Hurwitz would not apply.

Example 3.5. Let $f_n \in \mathcal{H}(\mathbb{D})$ be defined as

$$f_n(z) = \frac{1}{z+n}.$$

We know from Example 2.12 that these functions form a normal family and it follows that these functions (or a subsequence) converge compactly on $\mathbb{D}$ to some $f(z)$. Observing that the functions are monotone decreasing at each fixed z, we do not need to pass to a subsequence. These functions are never equal to 0 on $\mathbb{D}$ so by Hurwitz's theorem, the limit function must either be never zero or identically zero. fixing $z = 0$, we see that $1/n \to 0$ so the limit function satisfies $f(0) = 0$. Thus $f(z) \equiv 0$ everywhere on $\mathbb{D}$.

Example 3.6. Consider the sequence of functions $f_n \in \mathcal{H}(\mathbb{C})$ defined by

$$f_n(z) = e^z \cos\left(\frac{z}{n}\right).$$

For suitably large n, these functions are nowhere zero on any compact subset of $\mathbb{C}$. We first show the sequence is compactly convergent. The balls $\overline{B}(0, M)$, $M \in \mathbb{N}$ form an exhaustion of $\mathbb{C}$. Fixing some M, we will show normality on this ball. The image of any f_n at any point in $\overline{B}(0, M)$ is bounded by

$$\left|e^z \cos\left(\frac{z}{n}\right)\right| = |e^z| \cdot \left|\frac{e^{iz/n} + e^{-iz/n}}{2}\right| \leq e^M \cdot \frac{2e^M}{2} = e^{2M}.$$

Thus, Arzelà–Ascoli condition (ii) is satisfied. For equicontinuity, fix $\epsilon > 0$ and choose arbitrary $z, w \in \overline{B}(0, M)$. Observe that

$$\begin{aligned} |f_n(z) - f_n(w)| &= |e^z \cos(z/n) - e^w \cos(w/n)| \\ &\leq |e^z \cos(z/n) - e^w \cos(z/n)| + |e^w \cos(z/n) - e^w \cos(w/n)| \\ &\leq |e^z - e^w| + |e^w||\cos(z/n) - \cos(w/n)|. \end{aligned} \tag{3.1}$$

e^z is continuous and thus uniformly continuous on any compact set, and also bounded by e^M on $\overline{B}(0, M)$. It follows that we need only prove the equicontinuity of $\cos(z/n)$. This is because if have equicontinuity of these on $\overline{B}(0, M)$, then we can find a suitable $\delta_1 > 0$ for which

$$|\cos(z/n) - \cos(w/n)| < \frac{\epsilon}{2e^M}.$$

Then, combining this with the uniform δ_2 for $|e^z - e^w| < \epsilon/2$ here gives equicontinuity. By uniform continuity of $\cos(z)$, we can find $\delta_1 > 0$ such that for all $z, w \in \overline{B}(0, M)$,

$$|z - w| < \delta_1 \implies |\cos(z) - \cos(w)| < \frac{\epsilon}{2e^M}.$$

But now for all $n \in \mathbb{N}$, for all $z'/n, w'/n \in \overline{B}(0, M)$ the above holds. By translation, noting that $\overline{B}(0, M) \subset \overline{B}(0, nM)$, we have for all $n \in \mathbb{N}$ and $z', w' \in \overline{B}(0, M)$

$$|z' - w'| < \delta_1 \implies |\cos(z'/n) - \cos(w'/n)| < \frac{\epsilon}{2e^M}.$$

Now by (3.1), for any $n \in \mathbb{N}$ and all $z, w \in \overline{B}(0, M)$ we have

$$|f_n(z) - f_n(w)| \leq |e^z - e^w| + |e^z||\cos(z/n) - \cos(w/n)| < \epsilon/2 + e^M \frac{\epsilon}{2e^M} = \epsilon.$$

This gives equicontinuity of f_n over any element of the exhaustion, so we have equicontinuity anywhere in $\mathbb{C}$ giving Arzelà–Ascoli condition (i). It follows that the sequence is compactly convergent, and so by Hurwitz's theorem the limit function is either everywhere zero or nowhere zero. It is clear the limit is e^z, as fixing any z, the pointwise limit of the cosine approaches 1. This confirms what Hurwitz's theorem tells us, as e^z is nowhere zero as required.

3.2 Local boundedness and Montel's theorem

If we ask for the additional structure that our functions are holomorphic, it is natural to expect that sufficient conditions for normality may be stronger. This strength comes in the form of Montel's theorem, proven in his first paper on the subject in 1907 [21]. It turns out that for holomorphic families, we can remove the requirement of equicontinuity if we strengthen Arzelà–Ascoli condition (ii) to a local rather than pointwise property.

For the space $\mathcal{C}(\Omega, \mathbb{C})$, Arzelà–Ascoli condition (ii) is equivalent to the family being uniformly bounded at each $z \in \Omega$ by some constant $M_z > 0$, such that $|f(z)| \leq M_z$ for all $f \in \mathcal{F}$. This follows from the Heine-Borel theorem, as any bounded subset of $\mathbb{C}$ has compact closure. Instead of being pointwise bounded in this fashion we now define a stronger, local version of boundedness for a family.

Definition 3.7. A family $\mathcal{F} \subset \mathcal{H}(\Omega)$ is *locally bounded* on Ω if for all compact $X \subset \Omega$ there exists a constant $M_X > 0$ such that for all $f \in \mathcal{F}$ and $z \in X$, $|f(z)| \leq M_X$. A trivial open covering compactness argument tells us this is equivalent to being uniformly bounded on some open neighbourhood of each point in Ω.

Theorem 3.8 (Montel's Theorem)**.** *[1] Let $\Omega \subset \mathbb{C}$ be open and $\mathcal{F} \subset \mathcal{H}(\Omega)$, a family of holomorphic functions. Then $\mathcal{F}$ is normal if and only if it is locally bounded.*

Proof. First, suppose $\mathcal{F}$ is normal. Then by Arzelà–Ascoli, $\mathcal{F}$ is bounded and equicontinuous at each point in Ω. Let $X \subset \Omega$ be compact. Fix $\epsilon = 1$, by equicontinuity and Proposition 2.8 there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$, $z \in X$ and $w \in B(z, \delta)$,

$$|f(z) - f(w)| < 1.$$

By pointwise boundedness, for each $z \in X$ there exists $M_z > 0$ such that $|f(z)| \leq M_z$ for all $f \in \mathcal{F}$. Thus for $w \in B(z, \delta)$

$$|f(w)| \leq |f(z)| + |f(z) - f(w)| \leq M_z + 1.$$

By compactness the collection of open δ balls form an open cover of X so for a finite collection of $\{z_i\} \subset X$ we have

$$X \subset \bigcup_{i=1}^{n} B(z_i, \delta).$$

And thus for all $f \in \mathcal{F}$ and $w \in X$,

$$|f(w)| \leq \max_{1 \leq i \leq n} \{M_{z_i}\} + 1$$

so $\mathcal{F}$ is locally bounded.

To prove the converse, assume $\mathcal{F}$ is locally bounded. Then $\mathcal{F}$ is clearly pointwise bounded so condition (ii) of Arzelà–Ascoli holds. Thus, it is sufficient for us to show $\mathcal{F}$ is pointwise equicontinuous, then by Proposition 2.8 and Arzelà–Ascoli normality follows.

Fix $z \in \Omega$. Since Ω is open, we can find $r > 0$ such that $\overline{B}(z, 2r) \subset \Omega$. By local boundedness, let M be such that $|f(w)| \leq M$ for all $f \in \mathcal{F}$ and all $w \in \overline{B}(z, 2r)$. By Cauchy's integral formula, for any $z_1, z_2 \in \overline{B}(z, r)$, the concentric disc of half radius we have for any $f \in \mathcal{F}$:

$$\begin{aligned} f(z_1) - f(z_2) &= \frac{1}{2\pi i} \int_{|\xi - z| = 2r} \left(\frac{1}{\xi - z_1} - \frac{1}{\xi - z_2} \right) f(\xi) d\xi \\ &= \frac{z_1 - z_2}{2\pi i} \int_{|\xi - z| = 2r} \frac{f(\xi) d\xi}{(\xi - z_1)(\xi - z_2)}. \end{aligned}$$

We are integrating around a simple closed curve so this integral is bounded by the length of the contour multiplied by the supremum of the integrand over the curve[1]. We know that the length of the curve is $4\pi r$ and that for all $f \in \mathcal{F}$ we have $f(\zeta) \leq M$. Also observe that

[1]A standard approximation result for integrals, sometimes known as the estimation lemma or ML inequality.

$|\xi - z_i| > r$ for each z_i, since ξ runs over the boundary of the $2r$-ball and the z_i are inside the r-ball.

$$
\begin{aligned}
|f(z_1) - f(z_2)| &\leq \left(\frac{|z_1 - z_2|}{2\pi}\right)(4\pi r)\left(\sup_{|\xi - z| = 2r} \frac{|f(\xi)|}{|\xi - z_1||\xi - z_2|}\right) \\
&\leq 2r|z_1 - z_2| \quad \frac{M}{r^2} \\
&= \frac{2M}{r}|z_1 - z_2|.
\end{aligned}
$$

Now for fixed $\epsilon > 0$ we choose

$$\delta = \frac{r\epsilon}{4M}.$$

Since M and r are independent of the choice of $f \in \mathcal{F}$ such that for all $f \in \mathcal{F}$ we have $|f(w_1) - f(w_2)| \leq \epsilon/2 < \epsilon$ whenever $|w_1 - w_2| < \delta$. Thus $\mathcal{F}$ is pointwise equicontinuous, so equicontinuous on compacta by Proposition 2.8. We have established both conditions (i) and (ii) for Arzelà–Ascoli, so $\mathcal{F}$ is normal. □

In essence, Montel's theorem is saying that the space $\mathcal{H}(\Omega)$ has the Heine-Borel property that this form of boundedness along with closure is sufficient for compactness. Note that a family being locally bounded is not equivalent to being bounded in the metric ρ however, as ρ is always bounded by 1. Remember that we could only apply Arzelà–Ascoli here because $\mathcal{H}(\Omega)$ is a complete metric space.

We demonstrate the utility of Montel's theorem in the examples below, now no longer having to check equicontinuity as we did in the previous examples of continuous functions.

Example 3.9. Consider the family $\mathcal{F} \subset \mathcal{H}(\mathbb{D} \setminus \{0\})$ on the punctured disc given by

$$\mathcal{F} = \left\{ f(z) = \frac{1}{z^t} : t \in (1,2) \right\}.$$

For any $X \subset \mathbb{D} \setminus \{0\}$ compact, by closure, the infimum radial distance $r > 0$ from any point in X to 0 is attained by some $z = re^{i\theta} \in X$, so the points are bounded away from 0 on X. Thus on X, we have

$$|f(z)| = \frac{1}{|z|^t} \leq \frac{1}{|z|} \leq \frac{1}{r}$$

for all $f \in \mathcal{F}$, so $\mathcal{F}$ is locally bounded on the punctured disc, and thus normal here.

Example 3.10. Consider the family $\mathcal{F} \subset \mathcal{H}(\mathbb{D})$ given by

$$\mathcal{F} = \{f(z) = \sin(nz) : n \in \mathbb{N}\}.$$

We saw a similar family already in Example 2.11 where instead the domain of the functions was $(0,1)$. There $\mathcal{F}$ failed to be normal, as although all the values of the functions lay inside the compact $[-1,1]$, it was not equicontinuous at any point in $(0,1)$ (we only showed one place as this was enough to establish lack of normality).

It is clear that for compact subsets of $(0,1) \subset \mathbb{R}$, the functions are locally bounded and since these functions are holomorphic on $(0,1)$, one might conclude this family is normal on $(0,1)$ by Montel's theorem, contradicting our prior result. However, this argument makes a subtle oversight. Although the interval $(0,1)$ is open in $\mathbb{R}$, it is *not* open in $\mathbb{C}$ so we cannot invoke Montel's theorem to deduce normality here. A crucial part of our proof involved taking an open neighbourhood of each point, and each point in $(0,1)$ does not have an open neighbourhood inside this set when viewed as a subset of $\mathbb{C}$.

Returning to the family $\mathcal{F}$ here, we see that it is not locally bounded on $\mathbb{D}$. Although it is on the real line, the holomorphic extension of $\sin(z)$ to $\mathbb{C}$ is unbounded. Assuming such a bound M did exist on the compact $\{i/2\} \subset \mathbb{D}$ would imply

$$|\sin(ni/2)| = \left|\frac{e^{n/2} - e^{-n/2}}{2}\right| \leq M \quad \text{for all } n \in \mathbb{N}.$$

This is clearly false as this modulus can be made arbitrarily large by choosing n large enough. It follows that $\mathcal{F}$ fails to be normal on $\mathbb{D}$ by Montel's theorem.

Example 3.11. Consider the family of holomorphic functions $\mathcal{F} \subset \mathcal{H}(\mathbb{C})$ given by

$$\mathcal{F} = \{f(z) \equiv n \mid n \in \mathbb{N}\}.$$

These functions are each well behaved and any sequence must either have infinitely many terms the same, in which case there is a convergent subsequence or alternatively the sequence must tend to infinity. The way in which these sequences approach infinity is uniform, so in a certain sense this family feels somewhat akin to a normal family. However, this family is clearly not locally bounded and thus not normal under our current definition. We will discuss this case and how to deal with this convergence to infinity in Chapter 5.

We have now developed sufficient machinery to discern the normality of many families of continuous and holomorphic functions, and are ready to put this to use to prove the Riemann mapping theorem.

4 The Riemann Mapping Theorem

In 1851 Bernhard Riemann claimed the following (translated from German) in his inaugural dissertation. *"Two given simply connected plane surfaces can always be mapped onto one another in such a way that each point of the one corresponds to a unique point of the other in a continuous way and the correspondence is conformal; moreover, the correspondence between an arbitrary interior point and an arbitrary boundary point of the one and the other may be given arbitrarily, but when this is done the correspondence is determined completely."* [14]

This was the first formulation of the Riemann mapping theorem. His proof of the result, however, was flawed - it made excessive assumptions regarding the smoothness of the boundary, as well as the existence of a particular minimum when in fact all that could be asserted was that the infimum was non-negative. The first rigorous proof of the result was given by William Osgood nearly 50 years later in 1900 [36]. The proof we give here is different to both original approaches, making use of the machinery regarding normal families we have developed so far. Crucial to the proof are the theorems of Hurwitz and Montel, as well as another powerful lemma that we will prove. We now state the theorem in its modern form.

4.1 Statement and preparation

Theorem 4.1 (The Riemann Mapping Theorem)**.** *[1, 32, 35] Let $\Omega \subset \mathbb{C}$ be a simply connected domain which is not all of $\mathbb{C}$. Then there exists a biholomorphism $F : \Omega \to \mathbb{D}$. If we further choose a point $z_0 \in \Omega$, and require $F(z_0) = 0$ and $F'(z_0) > 0$ then this biholomorphism is unique.*

Topological equivalence between simply connected domains in $\mathbb{C}$ is of course trivial, but the fact that we must have a biholomorphism is not at all evident at first. Note that there can be no such biholomorphism for $\Omega = \mathbb{C}$ by Liouville's theorem. Such a biholomorphism from $\mathbb{C}$ to $\mathbb{D}$ would be entire and bounded and thus necessarily constant. We do *not* need the domain to be bounded, however, as we see with Ω_2 in Figure 4.1.

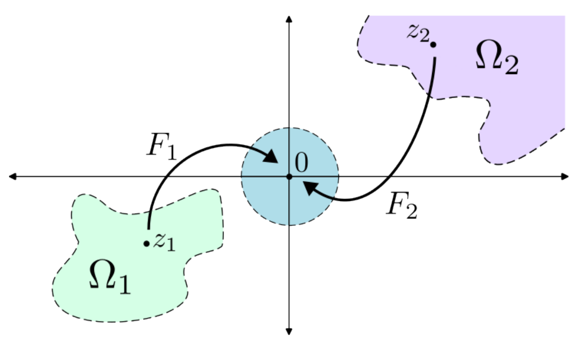

Figure 4.1: Simply connected domains Ω_1, Ω_2, which must have biholomorphisms to $\mathbb{D}$.

Corollary 4.2. *Any two simply connected domains are biholomorphically equivalent.*

Proof. This follows immediately, as if Ω_1, Ω_2 are arbitrary simply connected domains then there exist biholomorphisms $F_1 : \Omega_1 \to \mathbb{D}$ and $F_2 : \Omega_2 \to \mathbb{D}$ and thus the composition $F_1 \circ F_2^{-1} : \Omega_2 \to \Omega_1$ is a biholomorphism between these domains. □

If there is one biholomorphism from Ω to $\mathbb{D}$ then we could generate an infinite number by subsequently composing with any arbitrary disc rotation. By requiring that $F'(z_0)$ is real and positive this gives us a unique choice from the equivalence class of biholomorphisms equivalent under rotation. Thus fixing the argument of an image point is a necessary condition for our uniqueness claim, and it turns out it will also be sufficient.

Before we begin our proof of the Riemann mapping theorem, we establish a powerful lemma below.

Lemma 4.3 (Schwarz's lemma)**.** *[27, 35] Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic with $f(0) = 0$. Then*

$$|f(z)| \leq |z|$$

for all $z \in \mathbb{D}$, and $|f'(0)| \leq 1$. Furthermore, if $|f'(0)| = 1$ or $|f(z)| = |z|$ for some non zero $z \in \mathbb{D}$ then we must have $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Proof. Assume f is as in the theorem, and consider the function $g : \mathbb{D} \to \mathbb{D}$ given by:

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } 0 < |z| < 1 \\ f'(0) & \text{if } z = 0 \end{cases}$$

We claim $g(z)$ is holomorphic on $\mathbb{D}$. Clearly, g is holomorphic on $\mathbb{D} \setminus \{0\}$, as it is the quotient of two holomorphic functions and the denominator is non-zero. To see g extends to be holomorphic at $z = 0$, consider the Taylor series for $f(z)$ which exists by holomorphicity:

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} a_n z^n; \quad a_n \in \mathbb{C} \\ &= a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \ldots \\ &= z(a_1 + a_2 z + a_3 z^2 + a_4 z^3 \ldots). \end{aligned}$$

Note that the sum starts from $n = 1$ since our condition $f(0) = 0 \implies a_0 = 0$. Clearly dividing by z to obtain a Laurent series for $f(z)/z$ gives a Taylor series, so $f(z)/z$ extends to be analytic and thus holomorphic at $z = 0$. Evaluating this power series at 0 gives $a_1 = f'(0)$ matching our definition for g as required; so g is indeed holomorphic on $\mathbb{D}$.

Now fixing some $r < 1$, for any z with $|z| < r < 1$ by the max modulus principle we have g bounded by the supremum of the values it takes on the unit circle:

$$|g(z)| \leq \sup_{|z|=r} |g(z)| = \sup_{|z|=r} \left| \frac{f(z)}{z} \right| \leq \frac{\sup_{|z|=r} |f(z)|}{r} \leq \frac{1}{r}.$$

Where in the last inequality we have used that f maps into $\mathbb{D}$ so is bounded by 1. Now noting that the LHS is independent of r, we take the limit $r \to 1^-$, giving us the required inequality:

$$|g(z)| \leq 1 \implies |f(z)| \leq |z|.$$

For the proof of the second condition, first, assume $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$. Then $|g(z_0)| = 1$, and since $|g(z)| \leq 1$ then by the max modulus principle, $g(z) = a$ for some $a \in \mathbb{C}$ with $|a| = 1$. It follows that $f(z) = az$. In the case that $f'(0) = 1$ this directly implies $g(0) = 1$, so by the max modulus principle again we deduce $f(z) = az$, $|a| = 1$ as required. □

4.2 Proof of the Riemann mapping theorem

We follow proof of the Riemann mapping theorem from [1, 32, 35]. Our proof will focus on the family of functions

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid \text{holomorphic, injective and } f(z_0) = 0\}, \tag{4.1}$$

where z_0 is the point given in the statement of the theorem. We proceed in 4 stages.

1. We will show that $\mathcal{F}$ is non-empty, by constructing a member.
2. Next, we prove $\mathcal{F}$ is normal in $\mathcal{H}(\Omega)$. This will be used to establish that there is an $F \in \mathcal{F}$ with maximal value of $|F'(z_0)|$.
3. We then deduce that this F surjects onto $\mathbb{D}$, establishing it as a biholomorhism.
4. Finally, we show that the conditions stated in the theorem force F to be unique.

4.2.1 Showing $\mathcal{F}$ is non-empty

We will first show that $\mathcal{F}$ is non-empty, by constructing a member of the family which satisfies all necessary conditions. This construction comes from [32].

Since $\Omega \neq \mathbb{C}$ we can find $a \in \mathbb{C} \setminus \Omega$. Since $z - a$ has no zero on Ω there exists $g \in \mathcal{H}(\Omega)$ with $g(z)^2 = z - a$ with some choice of branch of square root.

Let $z, w \in \Omega$. If $g(z) = g(w)$ or $g(z) = -g(w)$ then $z = w$, since

$$z - a = g(z)^2 = g(w)^2 = w - a.$$

So not only is g injective, but also if g takes some value $g(z)$ on Ω then it cannot also take the value $-g(z)$ on Ω, as the only candidate value that g could map to $-g(z)$ is z, but this is mapped to $g(z)$. By our construction with the holomorphic root, these values are never equal as this could only occur if they were both zero. Thus

$$g(\Omega) \cap (-g(\Omega)) = \emptyset. \tag{4.2}$$

Fix $w \in \Omega$. By the open mapping theorem, there must be some open ball in the image of Ω under g, $B(w, \epsilon) \subset g(\Omega)$ for some $\epsilon > 0$. By (4.2), $B(-w, \epsilon)$ is disjoint from $g(\Omega)$ so $|w + g(z)| \geq \epsilon$ for all $z \in \Omega$, see Figure 4.2.

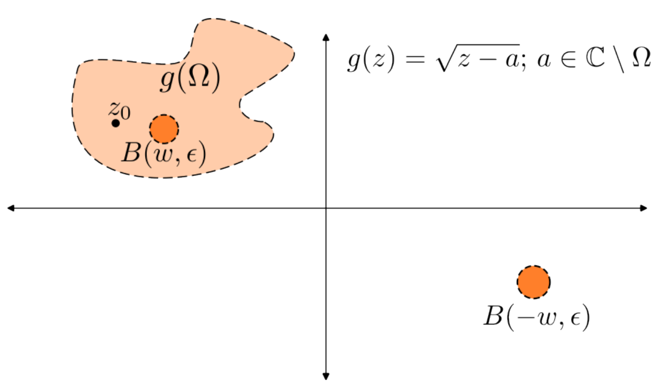

Figure 4.2: Finding an open ball $B(-w, \epsilon)$ disjoint from $g(\Omega)$.

We now construct a function based on g which satisfies all our requirements for membership of $\mathcal{F}$:

$$f(z) = \frac{\epsilon/3}{w + g(z)} - \frac{\epsilon/3}{w + g(z_0)}.$$

Holomorphicity and injectivity are clearly inherited from g, and we have $f(z_0) = 0$. For $z \in \Omega$ we have

$$\begin{aligned} |f(z)| &= \left| \frac{\epsilon/3}{w + g(z)} - \frac{\epsilon/3}{w + g(z_0)} \right| \\ &\leq \frac{\epsilon/3}{|w + g(z)|} + \frac{\epsilon/3}{|w + g(z_0)|} \\ &\leq \frac{\epsilon/3}{\epsilon} + \frac{\epsilon/3}{\epsilon} = 2/3 < 1, \end{aligned}$$

first by triangle inequality, then since $|w+g(z)| \geq \epsilon$ for $z \in \Omega$. Thus $f \in \mathcal{F}$, so $\mathcal{F}$ is non-empty as required.

Finally, we also observe that

$$f'(z) = \frac{-g'(z)}{(w + g(z))^2} > 0 \implies |f'(z_0)| > 0. \tag{4.3}$$

4.2.2 Existence of $F \in \mathcal{F}$ with maximal $|F'(z_0)|$

We will now make use of the theorems of Montel and Hurwitz to establish that there is some function $F \in \mathcal{F}$, such that the value $|F'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$. Since each function is holomorphic then the value of $|f'(z_0)|$ is well defined and finite for each f, but it is not apparent that the supremum of these values should be finite, let alone attained by some function in the family. We set c as this supremum with the knowledge that we could have $c = \infty$:

$$c = \sup \left\{ |f'(z_0)| : f \in \mathcal{F} \right\}.$$

We must have $c > 0$ since the function constructed in the previous section has a non-zero derivative at z_0. It is clear that there must be some sequence of functions $f_n \in \mathcal{F}$ for which $f_n(z_0) \to c$, if this were not the case then every sequence of this form would be bounded away from c from below and thus we could choose a new smaller bound for these values, contradicting that c is the supremum.

Since all $f \in \mathcal{F}$ map to $\mathbb{D}$, their values are bounded by $|f(z)| < 1$ uniformly, and thus by Montel's theorem $\mathcal{F}$ is a normal family. Thus there exists a convergent subsequence of the above sequence, $f_{n_k} \to F$ for some F. By uniform convergence F is holomorphic, $F(z_0) = 0$ and $|F(z)| \leq 1$ on Ω. By the open mapping theorem F must map open sets to open sets, and so we can further deduce that $|F(z)| < 1$, as F can at the most map to the largest open set inside the closed disc, the open disc $\mathbb{D}$. It remains to show injectivity of F, and this is established through an application of Hurwitz's theorem.

Fix $z_1 \in \Omega$ and consider the compactly convergent sequence of functions defined on Ω by

$$g_n(z) := f_n(z) - f_n(z_1).$$

with compactly convergent limit $g(z)$. Since $f_n(z)$ are injective, it follows that $g_n(z) \neq 0$ on $\Omega \setminus \{z_1\}$ as otherwise, $f_n(z)$ would attain the same value at z_1 as it does elsewhere, contradicting injectivity. Now by Hurwitz's theorem, the limit function $g(z) = F(z) - F(z_1)$ either is everywhere zero or never zero on Ω.

If g is everywhere zero then F is constant with $F(z) \equiv F(z_1) = F(z_0) = 0$, but then $F'(z_0) = c = 0$, implying all the functions in the family have zero modulus derivative at z_0. But this is not the case since our previous construction (4.3) has a positive modulus derivative at z_0. Thus it must be the case that $F(z) - F(z_1) \neq 0$ on $\Omega \setminus \{z_1\}$. Since we fixed z_1 arbitrarily, this establishes injectivity of F, since it cannot take the same value at any distinct $z, z_1 \in \Omega$.

F satisfies all the necessary conditions for membership of $\mathcal{F}$, so we must have $F \in \mathcal{F}$ which attains the maximal value of $|F'(z_0)|$ as required.

4.2.3 F surjects onto $\mathbb{D}$

We have now shown that $\mathcal{F}$ is non-empty and contains a function $F \in \mathcal{F}$ for which $|F'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$. We will now prove that this F is surjective onto $\mathbb{D}$, and thus we have found our biholomorphism from Ω to $\mathbb{D}$. This is established by showing that if F did not surject, we would be able to construct a function with a larger derivative at z_0, a contradiction to F having the maximal such value.

Assume that F does not surject onto $\mathbb{D}$, then there exists some $w \in \mathbb{D} \setminus F(\Omega)$. We define ψ_w as the disc automorphism which interchanges w and 0, given by

$$\psi_w(z) = \frac{w - z}{1 - \overline{w}z}.$$

We have that $\psi_w \circ F(z)$ is nowhere zero, since F maps no point to w, and the only point mapped to 0 by ψ_w is w. Thus we can define a holomorphic root of this function on Ω:

$$h(z) = \sqrt{\psi_w \circ F(z)}.$$

We have that $h(z_0) = \sqrt{\psi_w(0)} = \sqrt{w}$. We now compose h with $\psi_{\sqrt{w}}$, the disc automorphism which sends $\sqrt{w}$ to zero, and define this new function as

$$G(z) = \psi_{\sqrt{w}} \circ h(z).$$

We have $G(z_0) = \psi_{\sqrt{w}}(h(z_0)) = \psi_{\sqrt{w}}(\sqrt{w}) = 0$, and $G(z)$ is bijective as is a composition of bijective maps - F is a biholomorphism so bijective, both disc automorphisms are bijective, and since we have fixed a branch of square root this is too. We also know G is holomorphic, and maps $\Omega \to \mathbb{D}$ as each composition takes $\mathbb{D}$ to $\mathbb{D}$. Since G satisfies all conditions for membership we have $G \in \mathcal{F}$. We have that

$$F(z) = \psi_w \circ \mathrm{sq} \circ \psi_{\sqrt{w}} \circ G(z) = \Psi \circ G(z)$$

where

$$\Psi : \mathbb{D} \to \mathbb{D} \qquad \Psi(z) = \psi_w \circ \mathrm{sq} \circ \psi_{\sqrt{w}}(z) \qquad \mathrm{sq}(z) = z^2.$$

Now by Schwarz's lemma, $|\Psi'(z)| \leq |z|$ and $|\Psi'(0)| \leq 1$. If $|\Psi'(0)| = 1$ then by Schwarz's lemma $\Psi(z) = az$, $a \in \mathbb{C}$ with $|a| = 1$. This would imply Φ is injective. But we know this cannot be true since it is composed of the squaring function sq which is 2 to 1, and thus we cannot have $|\Psi'(0)| = 1$. So in fact $|\Psi'(0)| < 1$. Now by the chain rule

$$F'(z_0) = \Psi'(G(z_0))G'(z_0) = \Psi'(0)G'(z_0),$$

implying $|F'(z_0)| < |G'(z_0)|$, contradicting our maximality assumption for $|F'(z_0)|$. Thus it must be that no such $w \in \mathbb{D} \setminus F(\Omega)$ exists, so F surjects onto $\mathbb{D}$ as required.

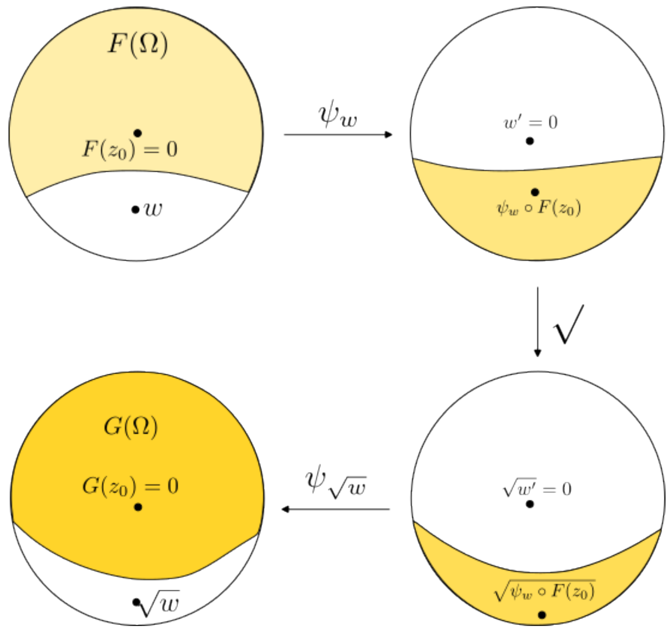

Figure 4.3: Constructing $G(z)$.

4.2.4 Uniqueness of the map

Finally, we prove the uniqueness claim with a simple application of Schwarz's lemma.

Let $F_1,\ F_2 : \Omega \to \mathbb{D}$ be two biholomorphisms as in the Riemann mapping theorem. We let $f := F_1 \circ F_2^{-1}$, and observe that

(a) f and f^{-1} are bijections $f, f^{-1} : \mathbb{D} \to \mathbb{D}$,
(b) both map $f(0) = f^{-1}(0) = 0$,
(c) both have real positive derivatives $f'(0), (f^{-1})'(0) > 0$.

By Schwarz's lemma on f and f^{-1} we have both

$$|f(z)| \leq |z| \text{ and } |f^{-1}(z)| \leq |z|.$$

Letting $w = f(z)$ combined with the second inequality gives $|z| \leq |f(z)|$, so in fact we must have that $|f(z)| = |z|$. By the second part of Schwarz's lemma, we have $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. Thus $f'(z) = a$ everywhere, so we must have $a = 1$ since the derivative is real and positive. We have concluded that $F_1 \circ F_2^{-1}(z) = f(z) = z$ and since both are bijections this gives us $F_1(z) = F_2(z)$ as required.

This completes the proof of the Riemann mapping theorem, as we have shown the existence of a unique biholomorphism F from an arbitrary simply connected open set $\Omega \subset \mathbb{C}$ to the unit disc $\mathbb{D}$ with the specifications stated. Note that this result tells us nothing about how one might go about finding such a mapping, merely its existence.

5 Meromorphic Normal Families

In Chapter 2 and Chapter 3, we developed our understanding of normal families of continuous and holomorphic functions, providing powerful characterizations of each in the form of the Arzelà–Ascoli theorem and Montel's theorem. We discussed in Example 3.11 that the convergence of the functions $f_n(z) \equiv n$ was well-behaved in the way it approached infinity. In this chapter, we extend the notion of normality we have defined to allow convergence to infinity in certain cases. This will enable us to construct a sensible notion of normality for families of meromorphic functions.

5.1 The Riemann sphere

We begin by equipping the codomain of meromorphic functions $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ with a metric. We achieve this by viewing points in $\mathbb{C}_\infty$ as points on the sphere S^2 and choosing a metric which suitably corresponds to the distance between the spherical representation of each point. Construction of the Riemann sphere by stereographic projection is seen in most first courses in complex analysis, so we briefly summarise the derivation here and make use of the formulas produced. For a detailed derivation, see [7, 8].

We will represent each point $z = x + iy \in \mathbb{C}$ as a point on the unit sphere in $\mathbb{R}^3$, the set of all points satisfying $r^2 + s^2 + t^2 = 1$. To avoid confusion with the complex number $z = x + iy$ we will use (r, s, t) for coordinates in $\mathbb{R}^3$. The representation of each $x + iy$ is given by the unique point on the sphere that intersects the line passing through the north pole of the sphere $N := (0, 0, 1)$ and the point $(x, y, 0)$, see Figure 5.1. Conversely, given any point on the

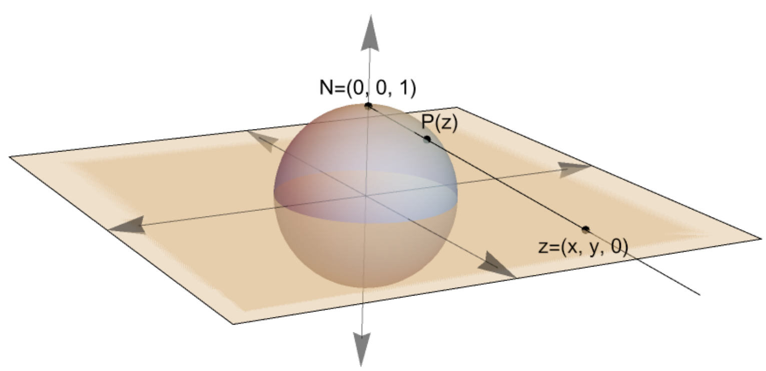

Figure 5.1: Inverse stereographic projection of the point $(x, y, 0)$ onto S^2.

sphere $\mathbf{p} \in S^2$ that is not N, it is evident that the line passing through $\mathbf{p}$ and N intersects a unique point in the rs-plane. This establishes a bijection between $\mathbb{C}$ and $S^2 \setminus \{0\}$, given by the stereographic projection formulas P, P^{-1} below.

$$P : \mathbb{C} \to S^2 \setminus \{N\}$$

$$P(z) = \left(\frac{2x}{1 + |z|^2}, \frac{2y}{1 + |z|^2}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \qquad P^{-1}(r, s, t) = \frac{r}{1 - t} + i \frac{s}{1 - t}.$$

We can see that the formulas for P and P^{-1} are clearly continuous and thus P is a bijective continuous map with continuous inverse, establishing a homeomorphism $\mathbb{C} \cong S^2 \setminus \{N\}$.

Observe that as $x^2 + y^2 = |z| \to \infty$, $P(z) \to (0,0,1)$ and for a point $\mathbf{p} \in S^2$ as $\mathbf{p} \to N$, $|P^{-1}(\mathbf{p})| \to \infty$, since

$$|P^{-1}(\mathbf{p})| = \frac{r^2}{(1-t)^2} + \frac{s^2}{(1-t)^2} = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t} \to \infty \text{ as } t \to 1.$$

Thus by identifying infinity in all directions of $\mathbb{C}$ into a single "point at infinity", we extend the map P by $P(\infty) := (0,0,1) = N$, such that P defines a bijection

$$P : \mathbb{C}_\infty \to S^2.$$

Crucially, this is *not* a homeomorphism yet, as we have not established any choice of topology for $\mathbb{C}_\infty$. However, this will become a homeomorphism if we choose to *define* our metric on $\mathbb{C}_\infty$ to be exactly that which makes P a homeomorphism, such that distance between two points in $\mathbb{C}_\infty$ is given precisely by the difference between their spherical representations on S^2. We define our metric for $\mathbb{C}_\infty$ in this fashion below, making use of the Euclidean metric in $\mathbb{R}^3$, with derivation from [8, 12].

Definition 5.1. We define the *chordal metric* $\chi(w_1, w_2)$ for $w_1, w_2 \in \mathbb{C}_\infty$ by considering the Euclidean distance between the spherical representation of each point on S^2. This is the straight line distance between two points, given by the length of the chord of the sphere between them. It is not any notion of a distance traversed along the surface of the sphere. Let $w_1 = x + iy \in \mathbb{C}$ and $w_2 = u + iv \in \mathbb{C}$, and let $W_1 = P(w_1), W_2 = P(w_2)$. The distance between these points can be calculated directly by brute force algebra, but we can first make the computation slightly simpler with a trick from [8], observing that points $W_1, W_2 \in S^2 \subset \mathbb{R}^3$ are unit vectors and so satisfy

$$\|W_1 - W_2\|^2 = (W_1 - W_2) \cdot (W_1 - W_2) = \|W_1\|^2 + \|W_2\|^2 - 2W_1 \cdot W_2 = 2 - 2W_1 \cdot W_2.$$

By making use of this property, and multiplying by the denominators of each of W_1, W_2, we deduce

$$\begin{aligned}
&\|W_1 - W_2\|^2(|w_1|^2+1)(|w_2|^2+1) \\
&\quad = 2(|w_1|^2+1)(|w_2|^2+1) - 2(4xu + 4yv + (|w_1|^2-1)(|w_2|^2-1)) \\
&\quad = 2|w_1|^2|w_2|^2 + 2|w_1|^2 + 2|w_2|^2 + 2 - 8xu - 8yv - 2 + 2|w_1|^2 + 2|w_2|^2 - 2|w_1|^2|w_2|^2 \\
&\quad = 4|w_1|^2 + |w_2|^2 - 8xu - 8yv \\
&\quad = 4x^2 + 4y^2 + 4u^2 + 4v^2 - 8xu - 8yv \\
&\quad = 4(x-u)^2 + 4(y-v)^2 \\
&\quad = 4|w_1 - w_2|^2.
\end{aligned}$$

Now we define $\chi(w_1, w_2)$ for $w_1, w_2 \in \mathbb{C}$ by

$$\chi(w_1, w_2) := \|W_1 - W_2\| = \frac{2|w_1 - w_2|}{\sqrt{1+|w_1|^2}\sqrt{1+|w_2|^2}}.$$

This gives us a formula for this chordal distance entirely in terms of the two complex numbers $w_1, w_2 \in \mathbb{C}$. Taking the limit as $|w_2| \to \infty$ gives

$$\chi(w_1, \infty) = \frac{2}{\sqrt{1+|w_1|^2}}.$$

This agrees with a simple calculation of the Euclidean distance between W_1 and N, the north pole. By symmetry, the same holds for $\chi(w_2, \infty)$ and taking the limit of the above expression again as $|w_1| \to \infty$ gives 0 which once again agrees with the Euclidean distance being zero as the second point approaches the north pole. This establishes the chordal distance between any two points in $\mathbb{C}_\infty$, given by

$$\chi(w_1, w_2) := \begin{cases} \frac{2|w_1 - w_2|}{\sqrt{1+|w_1|^2}\sqrt{1+|w_2|^2}} & \text{if } w_1, w_2 \in \mathbb{C} \\ \frac{2}{\sqrt{1+|w_1|^2}} & \text{if } w_1 \in \mathbb{C} \text{ and } w_2 = \infty \\ \frac{2}{\sqrt{1+|w_2|^2}} & \text{if } w_2 \in \mathbb{C} \text{ and } w_1 = \infty \\ 0 & \text{if } w_1, w_2 = \infty. \end{cases}$$

It is clear that this defines a metric on $\mathbb{C}_\infty$, since it is precisely *defined* as the distance between the representatives on $S^2 \subset \mathbb{R}^3$, and we know that the Euclidean metric in $\mathbb{R}^3$ is a metric. Thus all the properties of metrics are directly inherited from this by definition.

We make the following observations. Since S^2 is a closed and bounded subset of $\mathbb{R}^3$, it is compact, so the metric space $(\mathbb{C}_\infty, \chi)$ is a compact metric space. The metric is bounded by $\chi(w_1, w_2) \leq 2$ since no two points on the sphere can be further apart than the length of the diameter. Distances between two points on the sphere are invariant under rotations about the origin, and reflections in planes which bisect the sphere along its great circles[1], so χ is also invariant under these. The map $w \mapsto 1/w$ corresponds to such a reflection in the rs-plane, and so

$$\chi(w_1, w_2) = \chi(1/w_1, 1/w_2) \text{ for all } w_1, w_2 \in \mathbb{C}_\infty. \tag{5.1}$$

Here we allow $1/\infty = 0$ and $1/0 = \infty$ in the way which intuitively follows, such that

$$\chi(w, 0) = \chi(1/w, \infty) \text{ for all } w \in \mathbb{C} \setminus \{0\}.$$

This can also be easily verified algebraically via direct substitution. In order for us to consistently extend notions of convergence from $\mathbb{C}$ to $\mathbb{C}_\infty$, we must confirm that the metric χ is suitably consistent with the Euclidean metric when we are considering subsets of $\mathbb{C}$. Two metrics are said to be *equivalent* if they induce the same topology. This means that a set is open with respect to the Euclidean metric if and only if it is open with respect to χ. We saw an example of this in Lemma 2.2 where instead we showed that convergence in one metric was equivalent to convergence in the other. We establish the topologies align on $\mathbb{C}$ from [8].

Proposition 5.2. *A set $U \subset \mathbb{C}_\infty$ is open in $(\mathbb{C}_\infty, \chi)$ if and only if*

(i) $U \cap \mathbb{C}$ is open in the Euclidean metric.
(ii) If $\infty \in U$, then there exists some $r > 0$ such that $\{z \in \mathbb{C} : |z| > r\} \subset U$.

Proof. Here the first condition is asserting the topological equivalence on $\mathbb{C}$. The second clarifies that sets including ∞ are open in $\mathbb{C}_\infty$ if and only if they contain some Euclidean neighbourhood of ∞, such that, for example, $\mathbb{D} \cup \{\infty\}$ is not open in $\mathbb{C}_\infty$.

[1] Circles on the surface of the sphere with maximal radius, such that their radius is shared with that of the sphere. Here, these are all the circles given by the sphere intersected with planes which include the origin $(0, 0, 0)$, since this is the centre of the sphere.

Condition (i) clearly follows from the fact that $P|_{\mathbb{C}} : \mathbb{C} \to S^2 \setminus \{N\}$ is a homeomorphism. A set $U \subset \mathbb{C}_\infty$ is open with respect to χ if and only if the set $P(U)$ is open in S^2 as we have precisely defined our topology on $\mathbb{C}_\infty$ to align with this. It is clear that $P(U)$ is open in S^2 if and only if $P(U) \setminus \{N\}$ is open in $S^2 \setminus \{N\}$. Thus $P^{-1}|_{\mathbb{C}}(P(U) \setminus \{N\}) = U \cap \mathbb{C}$ is open in $\mathbb{C}$, as homeomorphisms map open sets to open sets.

If some $U \subset \mathbb{C}_\infty$ with $\infty \in U$ is open in $\mathbb{C}_\infty$ then it must contain some neighbourhood of ∞

$$B_\chi(\infty, \delta) = \{z \in \mathbb{C}_\infty \mid \chi(\infty, z) < \delta\} \subset U.$$

We can assume without loss of generality that $\delta < 2$, and thus in this ball

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}} < \delta \iff \frac{4}{\delta^2} < 1 + |z|^2 \iff |z| > \sqrt{\frac{4}{\delta^2} - 1}.$$

This final expression is real and positive due to requiring $\delta < 2$. Thus condition (ii) holds as

$$\left\{ z \in \mathbb{C} : |z| > \sqrt{\frac{4}{\delta^2} - 1} \right\} \subset U.$$

Now we assume conditions (i) and (ii) and deduce that U is open. Since $P|_{\mathbb{C}}$ is a homeomorphism, $P|_{\mathbb{C}}(U \cap \mathbb{C})$ is open in $S^2 \setminus \{N\}$. If $\infty \notin \mathbb{C}_\infty$ then we can directly deduce this is open in S^2 and thus U is open in $\mathbb{C}_\infty$ as required. Alternatively if $\infty \in U$ then there exists

$$\{z \in \mathbb{C} : |z| > r\} \subset U \implies B_\chi\left(\infty, \frac{2}{\sqrt{1 + r^2}}\right) \subset U$$

so a Euclidean neighbourhood of infinity implies a spherical neighbourhood as required. □

This establishes the topological equivalence of χ and the Euclidean metric on $\mathbb{C}$. It is interesting to note that in the above proof, the balls in χ directly corresponded to balls in the Euclidean metric. This is due to the fact that the map P takes circles to circles. As a direct result of this topological equivalence, we can always find open balls of each type inside one another. We state this explicitly as a lemma in a similar formulation to [7].

Lemma 5.3. *(a) If $a \in \mathbb{C}$, and $r > 0$ is given, there exits $\epsilon > 0$ such that $B_\chi(a, \epsilon) \subset B(a, r)$.*
(b) If $a \in \mathbb{C}$, and $\epsilon > 0$ is given, there exits $r > 0$ such that $B(a, r) \subset B_\chi(a, \epsilon)$.
(c) If $\epsilon > 0$ is given, there exists $r > 0$ such that $\{z \in \mathbb{C} : |z| > r\} \subset B_\chi(\infty, \epsilon)$.
(d) If $r > 0$ is given, there exists $\epsilon > 0$ such that $B_\chi(\infty, \epsilon) \subset \mathbb{C}_\infty \setminus \{z \in \mathbb{C} : |z| \leq r\}$.

Proof. The first two statements follow directly from Proposition 5.2 condition (i), as the respective balls are open in the other metric, and thus the point a must have an open neighbourhood around it (which can be made arbitrarily small to not include ∞ in the second case). The third and fourth conditions follow directly from Proposition 5.2 condition (ii). □

It can be seen that the first two conditions are an equivalent formulation of the topological equivalence of χ to the Euclidean metric on $\mathbb{C}$, as the collection of open balls forms a basis for a topological space. This establishes all notions of convergence in either metric restricted to compact sets in $\mathbb{C}$ as equivalent, as being able to find a δ which works in one metric now directly allows us to find a δ which works in the other metric. We have now established machinery to rigorously deal with the point ∞ as if it were any other point and verified that it extends our previous formulations without contradiction. We are ready to begin dealing with notions of convergence for meromorphic functions.

5.2 The space of meromorphic functions

We have now established a metric suitable for dealing with functions taking values in $\mathbb{C}_\infty$. Our functions will now be of the form

$$f : (\Omega, |\cdot|_{\mathbb{R}^2}) \to (\mathbb{C}_\infty, \chi).$$

We extended our codomain with the goal of dealing with meromorphic functions and defining a sensible notion of normality for families of this type. Since the most general case we have seen was for continuous families in Chapter 2, we begin by confirming that meromorphic functions are *spherically continuous*. We will use the term *spherically* when referring to all notions of convergence and continuity of function in the metric χ. In this section we follow and elaborate on ideas from [1, 7, 11, 13, 29].

Proposition 5.4. *If a function f is meromorphic on domain Ω, then f is spherically continuous on Ω.*

Proof. We know a meromorphic function f is holomorphic except on a set of isolated points P_f, the poles of f. Fix $\epsilon > 0$ and let $z_0 \in \Omega$ be a point such that $f(z_0) \neq \infty$ and instead $f(z_0) = a$ for some $a \in \mathbb{C}$. By Lemma 5.3 there exists $r > 0$ such that $B(a, r) \subset B_\chi(a, \epsilon)$. Then, since holomorphic functions are continuous, there exists an $\delta > 0$ such that for all $z \in B(z_0, \delta)$. we have $f(z) \in B(a, r) \subset B_\chi(a, \epsilon)$. Thus f is spherically continuous at z_0.

Alternatively, we could have a pole at z_0, such that $f(z_0) = \infty$. By Lemma 5.3, there exists $r > 0$ such that $\{z \in \mathbb{C} : |z| > r\} \subset B_\chi(\infty, \epsilon)$. We know that the poles of a meromorphic function are isolated, and also that z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$. Due to this convergence, there must exit $\delta > 0$ such that for all $z \in B(z_0, \delta) \setminus \{z_0\}$, we have $|f(z)| > r$. Thus for $z \in B(z_0, \delta) \setminus \{z_0\}$ we have $f(z) \in \{z \in \mathbb{C} : |z| > r\} \subset B_\chi(\infty, \epsilon)$. For $z = z_0$, $f(z_0) = \infty \in B_\chi(\infty, \epsilon)$ and so for all $z \in B(z_0, \delta)$ we have $f(z) \in B_\chi(\infty, \epsilon)$. Thus f is spherically continuous at z_0. This concludes the proof, as we have shown that f is spherically continuous everywhere on Ω. $\square$

We are now ready to consider the space of meromorphic functions. We also redefine the space of holomorphic functions with this new metric.

$$\mathcal{M}(\Omega) := \{f : (\Omega, |\cdot|_{\mathbb{R}^2}) \to (\mathbb{C}_\infty, \chi) \mid f \text{ meromorphic}\}.$$

By the above proposition, $\mathcal{M}(\Omega) \subset \mathcal{C}(\Omega, \mathbb{C}_\infty)$. We once again use the metric $\rho(f, g)$ from Chapter 2, where in this case $(Y, d) = (\mathbb{C}_\infty, \chi)$. We can now also think of the space $\mathcal{H}(\Omega)$ by

$$\mathcal{H}(\Omega) := \{f : (\Omega, |\cdot|_{\mathbb{R}^2}) \to (\mathbb{C}_\infty, \chi) \mid f \text{ holomorphic}\}.$$

Such that $\mathcal{H}(\Omega) \subset \mathcal{M}(\Omega)$, with functions in $\mathcal{H}(\Omega)$ being meromorphic functions that never assume the value ∞. This is the same set we defined before but equipped with a different metric. It is still of the same form $\rho(f, g)$, but this is induced by χ rather than the Euclidean metric. We will call the specific case of convergence spherically uniformly on compact subsets of Ω *normal* convergence.

Definition 5.5. We say a sequence of meromorphic functions on Ω is *normally convergent* if it converges spherically uniformly on each compact subset $X \subset \Omega$. A family of meromorphic functions on Ω is *normal* if every sequence f_n has a normally convergent subsequence with limit $f \in \mathcal{M}(\Omega)$. This is often referred to as the *classical* definition of normality.

It is not at all clear that $\mathcal{M}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega, \mathbb{C}_\infty)$, as we do not know that normal convergence preserves meromorphicity. In fact, it does not as we see returning to the functions from Example 3.11.

Example 5.6. Let $\{f_n\} \subset \mathcal{M}(\Omega)$ be a sequence of functions defined by $f_n(z) \equiv n$ on Ω. Then for all $z \in \Omega$,

$$\chi(f_n(z), \infty) = \frac{2}{\sqrt{1+n^2}} \to 0 \text{ as } n \to \infty.$$

This implies f_n converges spherically uniformly to a constantly infinite function on Ω, and thus converges normally to this. This limit function is certainly not meromorphic. This was somewhat to be expected - considering ∞ as just another point in the codomain means that by our definitions, this really is just another continuous function in $\mathcal{C}(\Omega, \mathbb{C}_\infty)$. However, this could be a more general issue if the ways in which limit functions of holomorphic and meromorphic functions can assume infinity is not well-behaved.

Recall that we need our space of functions to be complete in order to apply Arzelà–Ascoli. Thus, we would like that the spaces $\mathcal{H}(\Omega)$ and $\mathcal{M}(\Omega)$ are complete metric spaces. Although the above example shows this not to be the case, it turns out the limit function being constantly infinite is the worst that can happen.

In a similar fashion to Hurwitz's theorem restricting the way in which compactly convergent holomorphic limit functions can assume a certain value, we will now prove a theorem which restricts how normally convergent meromorphic limit functions can assume the value ∞. We first observe a simple fact due to the inversion property (5.1) of the metric χ.

Lemma 5.7. *Let $\{f_n\} \subset \mathcal{M}(\Omega)$ be a sequence of meromorphic functions. Then $f_n \to f$ normally on Ω if and only if $1/f_n \to 1/f$ normally on Ω.*

Proof. For all $n \in \mathbb{N}$ and $z \in \Omega$, by the inversion property (5.1), we have that $\chi(f_n(z), f(z)) = \chi(1/f_n(z), 1/f(z))$. Thus any δ which gives spherical uniform convergence on some compact $X \subset \Omega$ for f_n also gives spherical uniform convergence here for $1/f_n$. The same applies in the other direction equivalently. □

Theorem 5.8. *[1, 7, 11] Let $\{f_n\} \subset \mathcal{M}(\Omega)$ be a normally convergent sequence of meromorphic functions with limit $f_n \to f$. Then either $f \in \mathcal{M}(\Omega)$, or $f(z) \equiv \infty$ on Ω.*
Let $\{f_n\} \subset \mathcal{H}(\Omega)$ be a normally convergent sequence of holomorphic functions with limit $f_n \to f$. Then either $f \in \mathcal{H}(\Omega)$, or $f(z) \equiv \infty$ on Ω.

Proof. Consider f_n meromorphic, converging normally to f. We know that we must at least have $f \in \mathcal{C}(\Omega, \mathbb{C}_\infty)$, as normal convergence is a form of compact convergence and this preserves continuity. Assume f is not identically ∞ and let z_0 be a point at which f is finite, with $|f(z_0)| = M$. We will first establish that there is some neighbourhood around z_0 on which f is holomorphic. By Lemma 5.3, let $\epsilon > 0$ be such that

$$B_\chi(f(z_0), \epsilon) \subset B(f(z_0), M). \tag{5.2}$$

By convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\chi(f_n(z_0), f(z_0)) < \epsilon/2$. The family

$$\{f, f_1, f_2, ...\} \subset \mathcal{C}(\Omega, \mathbb{C}_\infty)$$

is normal and therefore equicontinuous by Arzelà–Ascoli. Thus there exists $\delta > 0$ such that for $z \in B(z_0, \delta)$, we have $\chi(f_n(z), f_n(z_0)) < \epsilon/2$. By the triangle inequality, for $z \in B(z_0, \delta)$ and $n \geq N$ we have

$$\chi(f_n(z), f(z_0)) \leq \chi(f_n(z), f_n(z_0)) + \chi(f_n(z_0), f(z_0)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $f_n(z)$ is inside the χ-ball from (5.2), so by the triangle inequality in the Euclidean metric, for all $z \in \overline{B}(z_0, \delta)$ and $n \geq N$,

$$|f_n(z)| \leq |f_n(z) - f(z_0)| + |f(z_0)| \leq 2M. \tag{5.3}$$

By the formula for the metric χ, for z and n as above we have

$$\frac{2|f_n(z) - f(z)|}{\sqrt{1 + (2M)^2}\sqrt{1 + (2M)^2}} = \frac{2}{1 + 4M^2}|f_n(z) - f(z)| \leq \chi(f_n(z), f(z)),$$

where we have maximised the denominator such that the left expression is bounded above by the value of χ. Thus $\chi(f_n(z), f(z)) \to 0$ uniformly implies $|f_n(z) - f(z)| \to 0$ uniformly on this ball, so we have Euclidean convergence uniformly on $\overline{B}(z_0, \delta)$. By (5.3) the tail end of the sequence of f_n is bounded on this ball, so they must be holomorphic near z_0 for suitably large $n \geq N$. Thus by Euclidean compact convergence the limit function f must also be holomorphic on this neighbourhood of z_0. Since z_0 was chosen arbitrarily, it follows that the limit function is holomorphic on a neighbourhood of each point $z \in \Omega$ at which it is finite, and thus f is holomorphic on the subset of Ω for which $|f(z)| < \infty$.

Now consider a point $z_0 \in \Omega$ where $f(z_0) = \infty$. We know $1/f_n \to 1/f$ normally, and from the above argument, $1/f$ is holomorphic on the subset of Ω for which $|1/f(z)| \leq \infty$. Thus $1/f$ is certainly holomorphic everywhere it is zero. Since the zeros of a holomorphic function must be isolated, the zeros of $1/f$ are isolated and thus so are the z values for which $f(z) = \infty$. Thus f is holomorphic except on a set of isolated points where it takes the value ∞. Since we have spherical convergence, we know these must be poles, implying f is meromorphic on Ω.

Now consider the second statement, with holomorphic f_n normally convergent to f. Observe that $1/f_n$ is nowhere zero on Ω, since this would contradict the holomorphicity of f_n. By Hurwitz's theorem, $1/f$ is nowhere zero or identically zero. If $1/f \equiv 0$ on Ω, then $f \equiv \infty$ on Ω. If $1/f \neq 0$ everywhere on Ω then $|f(z)| < \infty$ everywhere on Ω, so by the above argument f is holomorphic everywhere on Ω as required. □

Corollary 5.9. *The subspaces of $\mathcal{C}(\Omega, \mathbb{C}_\infty)$ given by $\mathcal{H}(\Omega) \cup \{\infty\}$ and $\mathcal{M}(\Omega) \cup \{\infty\}$ are complete metric spaces.*

Example 5.10. Returning to the functions from our motivating example, let $\mathcal{F} \subset \mathcal{M}(\Omega)$ be the family

$$\mathcal{F} = \{f_n(z) \equiv n \mid n \in \mathbb{N}\}.$$

We have shown these are normally convergent with limit $f_n \to f \equiv \infty$. However, $\mathcal{F}$ is *not* normal in $\mathcal{M}(\Omega)$, as the limit function $f \equiv \infty$ simply does not exist inside this space. $\mathcal{F}$ is normal viewed as a family in $\mathcal{M}(\Omega) \cup \{\infty\}$ or in $\mathcal{C}(\Omega, \mathbb{C}_\infty)$, as these contain the limit point. Although we need not contain the limit function in the family, it must be in the space the functions are considered in.

This is analogous to the set $B = \{(x, y) \in X : 0 < x^2 + y^2 < 1\}$ viewed as a subset of either $X = \mathbb{R}^2$ or $X = \mathbb{R}^2 \setminus \{0\}$. In the first case $X = \mathbb{R}^2$, $\overline{B}$ is the closed disc, which we know to be compact, with any open cover admitting a finite subcover. In the case $X = \mathbb{R}^2 \setminus \{0\}$ however, the closure $\overline{B}$ in X would not include 0, and none of our open covering sets can contain 0. Taking an infinite collection of balls all tangent around 0 as part of the cover would mean that any finite cover would miss some points very close to 0. Thus B is not relatively compact in this X.

The point 0 in this example has the same role as ∞ with $\mathcal{F}$. The limit point simply does not exist in $\mathcal{M}(\Omega)$, so $\mathcal{F}$ is not relatively compact in $\mathcal{M}(\Omega)$. With this in mind, when we talk about a family $\mathcal{F} \subset \mathcal{M}(\Omega)$, we will say $\mathcal{F}$ is normal if it is normal in the complete space $\mathcal{M}(\Omega) \cup \{\infty\}$.

5.3 Spherical derivatives

We now study the spherical derivative of a meromorphic function. This will be the central object used in the formulation of a normality condition for meromorphic families given in the next section. The spherical derivative is a powerful tool which can be used to calculate the spherical length of differentiable curves in $\mathbb{C}_\infty$. In this section, we make use of results from [7, 11].

We know that for a differentiable curve $\gamma \subset \mathbb{C}$ in the plane parametrized by $\gamma(t)$ for $0 \le t \le 1$, the Euclidean length of the curve is given by

$$L(\gamma) := \int_\gamma |dz| = \int_0^1 |\gamma'(t)| dt.$$

Furthermore, by the chain rule, the length of the image curve $f(\gamma)$ under holomorphic f is

$$L(f \circ \gamma) := \int_\gamma |f'(z)||dz| = \int_0^1 |f'(\gamma(t))||\gamma'(t)| dt.$$

We would like a similar formula for the spherical length of a path on the Riemann sphere. We derive an infinitesimal form of the chordal metric by considering the limit

$$\lim_{h\to 0} \frac{\chi(z+h,z)}{|h|} = \lim_{h \to 0} \frac{2|z+h-z|}{|h|\sqrt{1+(z+h)^2}\sqrt{1+z^2}} = \frac{2}{1+z^2}.$$

Since the surface of a sphere is locally Euclidean, as we consider a smaller and smaller distance on the sphere's surface, this is approximated arbitrarily well by the chordal distance. It follows that for a differentiable curve $\gamma \subset \mathbb{C}_\infty$ on the Riemann sphere, the spherical length is given by

$$L_S(\gamma) = \int_\gamma \frac{2}{1+z^2} |dz| = \int_0^1 \frac{2|\gamma'(t)|}{1+|\gamma(t)|^2} dt.$$

Example 5.11. Let γ be the circle along the real line in $\mathbb{C}_\infty$, passing through ∞, 1, -1 and 0, parametrized by

$$\gamma(t) = \log\left(\frac{t}{1-t}\right) \qquad \gamma'(t) = \frac{1}{t(1-t)} \quad \text{for} \quad 0 \le t \le 1.$$

Then we expect $L_S(\gamma)$ to be 2π as it is the circumference of a circle with radius 1. We see that this is indeed the case.

$$L_S(\gamma) = \int_0^1 \frac{2\left|\frac{1}{t(1-t)}\right|}{\left|1+\log^2\left(\frac{t}{1-t}\right)\right|} dt$$

Substituting

$$u = \log\left(\frac{t}{1-t}\right) \quad du = \frac{dt}{t(1-t)} \quad t \to 0 \implies u \to -\infty \quad t \to 1 \implies u \to \infty.$$

$$L_S(\gamma) = \int_{-\infty}^{\infty} \frac{2du}{1+u^2} = 2\arctan u\Big|_{-\infty}^{\infty} = 2\pi \text{ as required.}$$

Returning to our previous equation, by the chain rule, we see that the spherical length of an image curve $f(\gamma)$ under meromorphic f is given by

$$L_S(f \circ \gamma) := \int_\gamma \frac{2|f'(z)|}{1+|f(z)|^2}|dz| = \int_0^1 \frac{2|f' \circ \gamma(t)||\gamma'(t)|}{1+|f \circ \gamma(t)|^2}dt.$$

It is not immediately clear that the integrand is defined where f has a pole, but we shall see that this is indeed the case. We define the spherical derivative $f^{\#}(z)$ of f to be such that

$$L_S(f \circ \gamma) := \int_\gamma 2f^{\#}(z)|dz|. \tag{5.4}$$

Definition 5.12. Let $f \in \mathcal{M}(\Omega)$. We define the *spherical derivative* of meromorphic f by

$$\begin{aligned} f^{\#}(z_0) :=& \frac{1}{2}\lim_{z\to z_0} \frac{\chi(f(z), f(z_0))}{|z-z_0|} \\ =& \frac{1}{2}\lim_{z\to z_0} \frac{\frac{2|f(z)-f(z_0)|}{|z-z_0|}}{\sqrt{1+|f(z)|^2}\sqrt{1+|f(z_0)|^2}} \\ =& \lim_{z\to z_0} \frac{|f'(z)|}{1+|f(z)|^2} \end{aligned}$$

Here we introduce the factor of $1/2$ to cancel with the leading constant 2 on the chordal metric. This is for simplicity of future calculations, as it is unnecessary clutter for our purposes[2].

It is clear to see that $f^{\#}$ is well-defined on the set of points for which f is holomorphic. However, if z_0 is a pole, then both f and f' approach ∞ in the limit. The limit for $f^{\#}$ is finite, however. To see this, recall that if f has a pole of order n at z_0, then we can find $\delta > 0$ such that $f(z)$ can be written as

$$f(z) = \frac{g(z)}{(z-z_0)^n} \quad g(z) \text{ holomorphic and } g(z) \neq 0 \text{ for } |z-z_0| < \delta.$$

On this punctured neighbourhood, f' in terms of g is given by

$$f'(z) = \frac{g'(z)(z-z_0)^n - g(z)n(z-z_0)^{n-1}}{(z-z_0)^{2n}}.$$

Returning to the spherical derivative we can see that

$$\begin{aligned} f^{\#}(z_0) =& \lim_{z\to z_0} \frac{|f'(z)|}{1+|f(z)|^2} \\ =& \lim_{z\to z_0} \frac{\left|g'(z)(z-z_0)^n - g(z)n(z-z_0)^{n-1}\right|}{|z-z_0|^{2n} + |g(z)|^2} \\ =& \frac{1}{|g(z_0)|} \in \mathbb{R}_{>0} \text{ if } n = 1 \text{ or } 0 \text{ if } n > 1, \end{aligned}$$

[2] In fact, many texts define the chordal metric without the leading 2 by instead projecting onto a sphere with radius $1/2$, centred around $(0, 0, 1/2)$. This was not done here for simplicity of derivation.

where we have used that $g(z) \neq 0$ to assert that the value is a positive real. Thus, $f^\#$ is well-defined and finite at each pole z_0, so is well-defined and finite on all Ω.

Proposition 5.13. *Let $f \in \mathcal{M}(\Omega)$. Then $f^\# = (1/f)^\#$ for all $z \in \Omega$ (this is the spherical derivative of $1/f$ and not the reciprocal of the spherical derivative).*

Proof. This is another consequence of the inversion property of the chordal metric. Looking at the definition of the spherical derivative, to obtain $(1/f)^\#(z_0)$ we would replace the numerator with $\chi(1/f(z), 1/f(z_0))$. But we know by (5.1) that this is equal to $\chi(f(z), f(z_0))$. □

In line with this, we extend our definition to include the spherical derivative of a constantly infinite function $f \equiv \infty$ by $f^\# = (1/f)^\# \equiv 0$, such that it behaves as any other constant function would.

Taking some $z = z_0 + \delta$, we see that $f^\#$ measures the change in spherical distance between the values of f from z to z_0. Taking $\delta \to 0$, therefore, gives us a measure of the "rate of expansion" of f on the surface of the Riemann sphere at z_0. Note that by the way we have defined this, it is a strictly non-negative quantity.

Lemma 5.14. *[11] Let $\{f_n\} \subset \mathcal{M}(\Omega)$ be a normally convergent sequence of meromorphic functions with limit $f_n \to f$. Then the sequence of spherical derivatives $f_n^\# \to f^\#$ converge compactly on Ω.*

Proof. Let $z \in \Omega$. By Theorem 5.8, at least one of f or $1/f$ must be holomorphic on some ball $\overline{B}(z, \delta) \subset \Omega$, since either f is meromorphic or $f \equiv \infty$. If f is holomorphic at z, then for some $N \in \mathbb{N}$ suitably large, all f_n are holomorphic here (and so are their derivatives). On this ball, $f_n \to f$ and $f_n' \to f'$ uniformly by topological equivalence. Thus the quotient

$$f_n^\# = \frac{|f_n'|}{1 + |f_n|^2} \to \frac{|f'|}{1 + |f|^2} = f^\#$$

uniformly also on this ball. Since z was arbitrary, then we have local uniform convergence and thus compact convergence of $f_n^\# \to f^\#$. Alternatively, if $1/f$ is holomorphic at z then the same argument with $(1/f_n)^\#$ gives $f_n^\# = (1/f_n)^\# \to (1/f)^\# = f^\#$ uniformly on every compact subset of Ω. □

5.4 Marty's theorem

We now prove an equivalent condition for the normality of meromorphic families, established by Frédéric Marty in his PhD thesis in 1931 under supervision from Montel [20]. As we discussed at the beginning of the chapter, $(\mathbb{C}_\infty, \chi)$ is a compact metric space and so any set is relatively compact, meaning that condition (ii) of Arzelà–Ascoli is trivially fulfilled for any $\mathcal{F}$. It follows that the normality of meromorphic families in the classical sense is equivalent to spherical equicontinuity, Arzelà–Ascoli condition (i). Note that as we discussed in Example 5.10, in order to apply Arzelà–Ascoli we must consider our families as from the larger, complete space $\mathcal{M}(\Omega) \cup \{\infty\}$. Equicontinuity is harder to check in the chordal metric, however. Marty showed that having locally bounded spherical derivatives is equivalent to equicontinuity and thus normality. The proof presented here is Marty's original proof and is very geometric in nature.

Theorem 5.15 (Marty's Theorem)**.** *A family $\mathcal{F} \subset \mathcal{M}(\Omega)$ is normal in $\mathcal{M}(\Omega) \cup \{\infty\}$ if and only if the family of spherical derivatives $\left\{f^\# : f \in \mathcal{F}\right\}$ is locally bounded.*

Proof. Assume the spherical derivatives are locally bounded. Let $z_0 \in \Omega$ and $r > 0$ be such that $\overline{B}(z_0, r) \subset \Omega$. Let $z_1 \in \overline{B}(z_0, r)$ and let γ be the differentiable curve given by the Euclidean line (in $\mathbb{C}$) between z_0 and z_1. Observe that the length of any image curve $f(\gamma) \subset \mathbb{C}_\infty$ between $f(z_0)$ and $f(z_1)$ for some $f \in \mathcal{F}$, viewed on the surface of the sphere S^2, is bounded below by the shortest Euclidean distance (in $\mathbb{R}^3$) between these two points, see Figure 5.2.

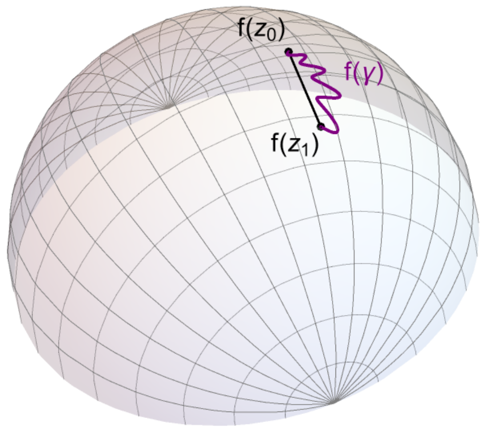

Figure 5.2: An image curve $f(\gamma)$ on the surface of the sphere.

This shortest Euclidean distance is given precisely by the chordal metric and thus we have

$$\chi(f(z_0), f(z_1)) \leq L_S(f \circ \gamma) = \int_\gamma 2f^\#(z)|dz|$$

By local boundedness of the derivatives on this ball, there exists $M > 0$ such that $2f^\#(z) \leq M$ for all $z \in \overline{B}(z_0, r)$ and $f \in \mathcal{F}$. This gives an upper bound for the integral by

$$\int_\gamma 2f^\#(z)|dz| \leq \int_\gamma M|dz| = M|z_0 - z_1|$$

$$\implies \chi(f(z_0), f(z_1)) \leq M|z_0 - z_1|$$

Since M is chosen independent of f, this gives equicontinuity at each $z_0 \in \Omega$ with $\delta = \epsilon/M$ for any $\epsilon > 0$ given. Thus $\mathcal{F}$ is normal by the Arzelà–Ascoli theorem.

To prove the converse, assume the spherical derivatives are not locally bounded. Then for some compact $X \subset \Omega$, we can find a sequence of functions $f_n(z) \in \mathcal{F}$ and a sequence of points $z_n \in X$ such that $f_n^\#(z_n) \to \infty$. It is clear that no subsequence of these spherical derivatives can converge in a Euclidean sense on X. Thus by the contrapositive to Lemma 5.14, $\mathcal{F}$ contains a sequence which does not admit a normally convergent subsequence and thus $\mathcal{F}$ is not normal. □

The similarity of Marty's theorem to Montel's theorem is evident - both relate normality to local boundedness in some way. Marty's theorem is more general, as it applies to any family of meromorphic functions not just holomorphic functions. We can of course use Marty's theorem on the space of holomorphic functions, but we must remember to make the space complete by considering $\mathcal{H}(\Omega) \cup \{\infty\}$. Montel's theorem only applies to $\mathcal{H}(\Omega)$ viewed with Euclidean metric on the codomain, which is already complete without ∞.

Example 5.16. Consider the family of meromorphic functions $\mathcal{F} \subset \mathcal{M}(\mathbb{D})$ given by

$$\mathcal{F} = \left\{ f(z) = \frac{1}{z^t} : t \in (1,2) \right\}.$$

These are the functions from Example 3.9, which have been extended to be meromorphic on $\mathbb{D}$. Consider the spherical derivative of a general $f \in \mathcal{F}$, given by

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2} = \frac{t|z|^{-t-1}}{1+|z|^{-2t}} = \frac{t|z|^{t-1}}{|z|^{2t}+1} \leq 2|z|^0 = 2.$$

Here we have used that $t < 2$, $|z|^{2t} + 1 \geq 1$ and $|z|^{t-1} \leq |z|^0$ for $z \in \mathbb{D}$ and $0 < t-1 < 1$. Thus the spherical derivatives are locally bounded by $|f^{\#}(z)| \leq 2$ on $\mathbb{D}$. By Marty's theorem, $\mathcal{F}$ is normal.

Example 5.17. Consider the family of holomorphic functions $\mathcal{F} \subset \mathcal{H}(\mathbb{D})$ given by

$$\mathcal{F} = \{ f(z) = nz : n \in \mathbb{N} \}.$$

We saw these functions defined on $(0,1)$ were not equicontinuous on $\mathbb{R}$ with the Euclidean metric in Example 2.7. We might hope for better behaviour in the spherical metric but this is not the case. The reason equicontinuity (and thus normality) failed in the Euclidean case was that $f(0) = 0$ for all functions, but for any $z \neq 0$, $|f(z)| \to \infty$ as $n \to \infty$. We see that in the meromorphic case, 0 causes a similar problem. Take any $X \subset \mathbb{D}$ compact which contains 0, for example $\{0\}$. A general spherical derivative is

$$f^{\#}(z) = \frac{n}{1+|nz|^2} \quad f^{\#}(0) = n,$$

so these can be made arbitrarily large by n at $z = 0$. By Marty's theorem, $\mathcal{F}$ is not normal.

6 General Conditions for Normality

We are now familiar with normality for families of continuous, holomorphic and meromorphic functions. In this chapter, we prove a powerful lemma due to Lawrence Zalcman, which allows us to formalise a heuristic principle for when certain properties of a family imply normality, known as Bloch's principle. We also make use of Zalcman's lemma to derive Montel's fundamental normality test for a family of holomorphic or meromorphic functions.

6.1 Normality preserving transforms of families

The following two lemmas are assumed without proof in many texts - that we can compose families with Möbius transforms while preserving normality. I have decided to formalise this and prove this further for any biholomorphism. These can both be somewhat intuited from the fact that normality is a local property, Corollary 2.10.

Lemma 6.1. *Let $\mathcal{F} \subset \mathcal{M}(\Omega)$ be a family of meromorphic functions, and let $h : D \to \Omega$ be a biholomorphism, $D \subset \mathbb{C}$ open. Then $\mathcal{F}$ is normal in $\mathcal{M}(\Omega) \cup \{\infty\}$ if and only if the family*

$$\mathcal{G} = \mathcal{F} \circ h := \{g = f \circ h \mid f \in \mathcal{F}\}$$

is normal in $\mathcal{M}(D) \cup \{\infty\}$.

Proof. We prove only one direction since the converse holds equivalently with the biholomorphism h^{-1}. Assume $\mathcal{F}$ is normal and let $z_0 \in D$ be an arbitrary point in the domain of $\mathcal{G}$, with $w_0 = h(z_0) \in \Omega$. We will show $\mathcal{G}$ is spherically equicontinuous at z_0, establishing normality (as we know Arzelà–Ascoli condition (ii) is trivially fulfilled in the chordal metric). Fix $\epsilon > 0$. By spherical equicontinuity of $\mathcal{F}$ at w_0, there exists $r > 0$ such that for all $f \in \mathcal{F}$ and all $w \in \Omega$

$$|w - w_0| < r \implies \chi(f(w), f(w_0)) < \epsilon.$$

Now by continuity of h, there exists $\delta > 0$ such that for all $z \in D$ with $|z - z_0| < \delta$ we have $|h(z) - h(z_0)| < r$. Since $h(z) \in \Omega$ for all $z \in D$, it follows that for all $f \in \mathcal{F}$ and all $z \in D$

$$|z - z_0| < \delta \implies \chi(f(h(z)), f(h(z_0))) < \epsilon,$$

giving spherical equicontinuity of $\mathcal{G}$ at z_0. It follows that $\mathcal{G}$ is normal in $\mathcal{M}(D) \cup \{\infty\}$. □

Each of the families of all meromorphic functions on some domain is the same family if viewed under the equivalence class of biholomorphisms on the domain, for any h as above. This lemma tells us that normality is an invariant of this equivalence. Thus, it is not the behaviour of specific functions on a specific region of $\mathbb{C}_\infty$ which gives rise to the normality of this class, but rather the disparity between how the functions themselves relate to each other on a generalised domain that is the equivalence class of biholomorphically equivalent domains.

We now prove a similar lemma, this time composing with a biholomorphism on the codomain of the functions. This is more complex, as we do not know that the values $f(z_0)$, $f \in \mathcal{F}$ are all in the same neighbourhood of the sphere for fixed z_0, so upon passing these values into a biholomorphism we would need uniform continuity of the biholomorphism to apply a composition continuity argument for equicontinuity at z_0 as in the previous lemma. Instead, we consider the images of general compact subsets of the domain itself and use this compactness to give uniform continuity of the biholomorphism. This combined with the normality of $\mathcal{F}$ will in turn help us show that every subsequence of the transformed family has a normally convergent subsequence. Inspiration for this proof comes from [19].

Lemma 6.2. *Let $\mathcal{F} \subset \mathcal{M}(\Omega)$ be a family of meromorphic functions with image inside $W \subset \mathbb{C}_\infty$ open, such that for all $f \in \mathcal{F}$ we have $f(\Omega) \subset W \subset \mathbb{C}_\infty$. Let $h : W \to D$ be a biholomorphism. Then $\mathcal{F}$ is normal in $\mathcal{M}(\Omega) \cup \{\infty\}$ if and only if the family*

$$\mathcal{G} = h \circ \mathcal{F} := \{g = h \circ f \mid f \in \mathcal{F}\}$$

is normal in $\mathcal{M}(\Omega) \cup \{\infty\}$.

Proof. As before, we only prove one direction as the converse is equivalent to using the biholomorphism h^{-1}. Assume $\mathcal{F}$ is normal. A general sequence in $\mathcal{G}$ is of the form $h \circ f_n$, $f_n \in \mathcal{F}$. By normality of $\mathcal{F}$, the sequence f_n has a normally convergent subsequence $f_{n_k} \to f$. We claim that the sequence $h \circ f_{n_k}$ converges normally to $h \circ f$.

Let $X \subset \Omega$ be compact and fix $\epsilon > 0$. Choose $r > 0$ such that the "thickening" of $f(X)$ by r, given by

$$f(X)_r := \{w \in \mathbb{C}_\infty \mid \chi(w, w_0) \leq r \text{ for some } w_0 \in f(X)\}$$

is contained by $f(X)_r \subset W$. X is compact in $\mathbb{C}$, so $f(X)$ is compact in $\mathbb{C}_\infty$ by continuity. W is open and contains the closed $f(X)$, so $f(X)$ is therefore bounded away from the boundary, allowing such an $r > 0$ to exist (or there is no boundary in the case $W = \mathbb{C}_\infty$, but in this case, it is trivially contained inside for any choice of r). Observe that since $f(X)$ is compact in $\mathbb{C}_\infty$, $f(X)_r$ is also. Since continuous functions on compact sets are uniformly continuous, there exists $\delta > 0$ such that for all $w_1, w_2 \in f(X)_r$

$$\chi(w_1, w_2) < \delta \implies \chi(h(w_1), h(w_2)) < \epsilon. \tag{6.1}$$

Now by normal convergence of f_{n_k}, there exists some $N \in \mathbb{N}$ such that for all $k \geq N$ and all $z \in X$, we have

$$\chi(f_{n_k}(z), f(z)) < \min\{\delta, r\}.$$

Thus for $k \geq N$ the images $f_{n_k}(X) \subset f(X)_r$. But now by (6.1) with $w_1 = f_{n_k}(z), w_2 = f(z)$ and the above inequality, for all $k \geq N$ and all $z \in X$ we have

$$\chi(h \circ f_{n_k}(z), h \circ f(z)) = \chi(h(f_{n_k}(z)), h(f(z))) < \epsilon.$$

This gives spherical uniform convergence $h \circ f_{n_k} \to h \circ f$ on an arbitrary compact set $X \subset \Omega$. Thus $h \circ f_{n_k} \to h \circ f$ normally on Ω, and $\mathcal{G}$ is normal in $\mathcal{M}(\Omega) \cup \{\infty\}$. □

Note that all the properties used in this proof could be significantly generalised. We only made use of the homeomorphic nature of h to deduce equivalence, we did not need its biholomorphic properties. Furthermore, the functions could have all come from $\mathcal{C}(\Omega, \mathbb{C}_\infty)$. We state and prove it in this format for the simplicity of our future application.

Corollary 6.3. *Let $\psi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transform. Then a family $\mathcal{F} \subset \mathcal{M}(\Omega)$ is normal in $\mathcal{M}(\Omega) \cup \{\infty\}$ if and only if the family $\psi \circ \mathcal{F}$ is.*

Example 6.4. Let $D \subset \mathbb{C}$ be a simply connected domain which is not all of $\mathbb{C}$, and let $\mathcal{F} \subset \mathcal{H}(\Omega)$ be all the functions of the form

$$\mathcal{F} := \{f \in \mathcal{H}(\Omega) \mid f : \Omega \to D\}.$$

By the Riemann mapping theorem, there exists a biholomorphism $h : D \to \mathbb{D}$. Then by Lemma 6.2, $\mathcal{F}$ is normal in $\mathcal{H}(\Omega) \cup \{\infty\}$ if and only if $h \circ \mathcal{F}$ is. But the family $h \circ \mathcal{F}$ maps to the unit disc so is trivially bounded and thus by Montel's theorem, $h \circ \mathcal{F}$ is normal viewed through the Euclidean metric. This implies normality of $h \circ \mathcal{F}$ in $\mathcal{H}(\Omega) \cup \{\infty\}$, so $\mathcal{F}$ is normal.

6.2 Bloch's heuristic principle

We now study a heuristic principle for the normality of a family, attributed to the French mathematician André Bloch. Bloch had a tragic life, fighting in World War 1 for several months before being injured and declared unfit to return to fighting. In 1917, at the age of 22, he murdered his brother, aunt and uncle. He was then confined to a psychiatric hospital from which he completed most of his contributions to mathematics, staying up to date on new research through correspondence with a number of academics on the outside [23]. The principle, roughly speaking, states that

"A family of holomorphic [meromorphic] functions which have a property P in common in a domain D is (apt to be) a normal family in D if P cannot be possessed by non-constant entire [meromorphic] functions in the finite plane." [15]

We see that this seems to align with Montel's theorem from Chapter 3. Consider the property

The family $\mathcal{F}$ is uniformly bounded on Ω.

We know by Liouville's theorem that no non-constant entire function can be bounded. Indeed by Montel's theorem, we know $\mathcal{F}$ is normal so the principle holds. However, it is ambiguous as stated. Rephrase the above as

Each function in $\mathcal{F}$ is bounded on Ω.

Once again, no non-constant entire function satisfies this, however the family $\mathcal{F} \subset \mathcal{H}(\mathbb{D})$

$$\mathcal{F} = \{f : \mathbb{D} \to \mathbb{C} \mid f(z) = nz\}$$

is not normal on $\mathbb{D}$ as we saw in Example 5.17. To formalise this principle, we need to be more precise in what a "property" can be. This was achieved in 1975 by Lawrence Zalcman, making use of a lemma which we now prove. Zalcman's lemma also provides further characterisation of normality. In this section, we follow the proof from [5, 11, 38].

Lemma 6.5 (Zalcman's Lemma)**.** *Let $\mathcal{F} \subset \mathcal{M}(\Omega)$ be a family of meromorphic [holomorphic] functions on a domain Ω. $\mathcal{F}$ is not normal if and only if there exist*

(i) a sequence of points $z_n \to z_0 \in \Omega$
(ii) a sequence of functions $f_n \in \mathcal{F}$
(iii) a sequence of real numbers $\rho_n > 0$ with $\rho_n \to 0$

such that the functions $g_n(\zeta) = f_n(z_n+\rho_n\zeta)$ converge normally to a non-constant meromorphic [entire] function $g(\zeta)$ on $\mathbb{C}$, satisfying $g^\#(\zeta) \leq 1$ for all $\zeta \in \mathbb{C}$.

Proof. First, assume $\mathcal{F}$ is not normal. Then by Marty's theorem, there exists a sequence of functions $\{f_n\} \subset \mathcal{F}$, some compact subset $X \subset \Omega$ and a sequence of points $w_n \to w \in X$ for which $f_n^\#(w_n) \to \infty$. By Lemma 6.1 we can assume without loss of generality[1] that $w_n \to 0 \in \Omega$ and that X is the closed disc $\{z : |z| \leq 1\} = \overline{\mathbb{D}} \subset \Omega$. By redefining w_n to be the tail end of the sequence we can assume it sits inside $\overline{\mathbb{D}}$ also.

Define

$$M_n := (1 - |z_n|^2) f_n^\#(z_n) = \max_{|z|\leq 1} (1 - |z|^2) f_n^\#(z).$$

Where $|z_n| \leq 1$ is the sequence of points which maximises this expression for each n. This maximum must exist since $\overline{\mathbb{D}}$ is compact, so there must be a point for which the supremum is attained. This must be finite for each M_n, since we know the spherical derivative is well-defined and finite on the entire domain, including at any poles of f. Observe that $M_n \to \infty$ since $f_n^\#(w_n) \to \infty$, so picking $z_n = w_n$ instead would give ∞ as a lower bound for the limit. Define

$$\rho_n := 1/f_n^\#(z_n) = \frac{1 - |z_n|^2}{M_n}. \tag{6.2}$$

We see that as $n \to \infty$, $\rho_n \to 0$ since $1 - |z_n|^2$ is bounded and $M_n \to \infty$.

$$0 \leq \rho_n \leq \frac{\rho_n}{1 - |z_n|^2} = \frac{1}{M_n} \to 0.$$

Observe also that $\rho_n > 0$, as $\rho_n = 0$ would imply $f_n^\#(z_n) = \infty$, which is impossible for a meromorphic function. We now have a sequence of points z_n, a sequence of functions $f_n \in \mathcal{F}$ and a sequence of reals $\rho_n > 0$ as specified. Consider the limit function created. Since all our functions f_n are defined on the closed disc $\overline{\mathbb{D}} \subset \Omega$, then the functions

$$g_n(\zeta) := f_n(z_n + \rho_n\zeta)$$

are defined when $|z_n + \rho_n\zeta| < 1$[2]. The argument of f_n is merely a linear transformation from $f_n(\zeta)$, so these g_n are meromorphic on $\mathbb{D}$. By the triangle inequality, we attain this when $|z_n| + \rho_n|\zeta| < 1$, giving

$$|\zeta| < \frac{1 - |z_n|}{\rho_n} = \frac{M_n}{1 + |z_n|} := R_n.$$

We can clearly see this $R_n \to \infty$ so the functions g_n are eventually defined on any compact subset of $\mathbb{C}$ for suitably large n. We fix ζ in some arbitrary compact set here by $|\zeta| \leq R < R_n$ for some R. We want to show that the spherical derivatives of the sequence $\{g_n\}$ are uniformly bounded on this general compact disc of radius R, then we can apply Marty's theorem to deduce the $\{g_n\}$ form a normal family. First, by the chain rule, we attain

$$g_n^\#(\zeta) = \frac{|g_n'(\zeta)|}{1 + |g_n(\zeta)|^2} = \frac{\rho_n |f_n'(z_n + \rho_n\zeta)|}{1 + |f_n(z_n + \rho_n\zeta)|^2} = \rho_n f_n^\#(z_n + \rho_n\zeta). \tag{6.3}$$

[1] Consider a biholomorphism which translates Ω so $w = 0$ then find a neighbourhood $B(0, r)$ and scale by a factor $2/r$ so the closed unit disc $\overline{\mathbb{D}}$ is contained inside.

[2] Of course, they are defined elsewhere in a transformed Ω but we do not know anything about Ω so we only make use of the behaviour on the disc.

Using this formulation of $g_n^\#(\zeta)$ along with (6.2) and the triangle inequality, we find

$$\begin{aligned} g_n^\#(\zeta) =& \frac{\rho_n M_n}{1-|z_n+\rho_n\zeta|^2} \\ =& \frac{1-|z_n|^2}{1-|z_n+\rho_n\zeta|^2} \\ \leq& \frac{1-|z_n|^2}{1-(|z_n|+\rho_n|\zeta|)^2} \\ =& \frac{1+|z_n|}{1+|z_n|+\rho_n|\zeta|}\cdot\frac{1-|z_n|}{1-|z_n|-\rho_n|\zeta|}. \end{aligned}$$

Here we can see that the first term is uniformly bounded above by 1 since the denominator is larger. For the second term, observe that for fixed R,

$$\frac{1-|z_n|}{1-|z_n|-\rho_n|\zeta|} \leq \frac{1-|z_n|}{1-|z_n|-\rho_n R} \to 1$$

as $n \to \infty$ since $\rho_n \to 0$. Thus we can uniformly bound the spherical derivatives $\{g_n^\#(\zeta)\}$ for $|\zeta| \leq R$. By Marty's theorem, $\{g_n\}$ is a normal family on $|\zeta| < R$. By replacing f_n with a suitable subsequence, we have that the sequence of $g_n(\zeta)$ is normally convergent to some limit $g(\zeta)$ defined on all of $\mathbb{C}$ since R was arbitrary. The g_n are either meromorphic or holomorphic, so Theorem 5.8 tells us either $g(\zeta)$ is also meromorphic or holomorphic respectively, or identically $g(\zeta) \equiv \infty$. By (6.3) and (6.2), the spherical derivative

$$g_n^\#(0) = \rho_n f_n^\#(z_n) = 1$$

independent of n so taking $n \to \infty$ gives $g^\#(0) = 1$, so g cannot be constant (or constantly infinity). Thus $g(\zeta)$ is a non-constant meromorphic (or holomorphic) function defined on all of $\mathbb{C}$. By the above inequalities, we have that for any fixed $R > 0$, $g^\#(\zeta) \leq 1$ for $|\zeta| \leq R$ so clearly we have $g^\#(\zeta) \leq 1$ for all $\zeta \in \mathbb{C}$.

The converse is considerably simpler. We assume $\mathcal{F}$ is normal and deduce that any such limit function must be constant. Indeed, suppose $\mathcal{F}$ is normal on Ω. Suppose there exist f_n, ρ_n and $z_n \to z_0 \in \Omega$ as stated in the theorem, with $g_n(\zeta) = f_n(z_n + \rho_n\zeta)$ converging normally to some $g(\zeta)$ meromorphic on all of $\mathbb{C}$. We can assume the points $z_n \in \overline{\mathbb{D}}$ by Lemma 6.1 and the fact that they converge. By Marty's theorem, there exists $M > 0$ such that for all $z \in \overline{\mathbb{D}}$ the spherical derivatives are bounded by $f_n^\#(z) \leq M$.

Fix $\zeta \in \mathbb{C}$, then for suitably large n we have $|z_n + \rho_n\zeta| \leq 1$. Then by (6.3),

$$g_n^\#(\zeta) = \rho_n f_n^\#(z_n + \rho_n\zeta) \leq \rho_n M.$$

Taking $n \to \infty$ gives $g^\#(\zeta) = 0$ as $\rho_n \to 0$. Since ζ was fixed arbitrarily, then g must have zero spherical derivative everywhere and thus g must be constant. □

Intuitively, Zalcman's lemma tells us that if we have a family of functions which is not normal on some domain Ω - and thus not normal at some point in Ω by Corollary 2.10 - then by appropriately zooming in on the region at which normality fails, the behaviour across all the functions in the family is "dynamic" enough that we can construct a non-constant meromorphic function on $\mathbb{C}$ out of the limit. Conversely, normal families are too "similar" at each point in the domain that stretching any local neighbourhood of the domain out to $\mathbb{C}$ forces any limit function to be constant.

We can see some connection of how this might connect to Bloch's principle, as it relates the normality of a family with if a non-constant entire function can be formed from the family. We are looking for a relation between if a non-constant function can possess a certain property and a family possessing a certain property being normal. The way to connect these two relations is not immediately obvious. We must first formalize the notion of a "property" that a family of meromorphic functions can satisfy. As established by Robinson in [26], it is useful to distinguish functions also by the domain they are defined on.

Definition 6.6. We define a *property* P of meromorphic functions as a set of tuples $\langle f, \Omega \rangle$ representing the function f with domain Ω. In this way, the elements $\langle f, \Omega \rangle$ and $\langle f, D \rangle$ with $D \neq \Omega$ are considered distinct functions, as they have different domains. We say a family $\mathcal{F}$ satisfies a property P on Ω if for all $f \in \mathcal{F}$ we have $\langle f, \Omega \rangle \in P$.

Theorem 6.7 (Bloch's Principle)**.** *Let P be a property of meromorphic functions satisfying the following:*

(i) If $\langle f, \Omega \rangle \in P$ and $D \subset \Omega$, then $\langle f, D \rangle \in P$.
(ii) If $\langle f, \Omega \rangle \in P$ and $\psi(z) = az + b$ then $\langle f \circ \psi, \psi^{-1}(\Omega) \rangle \in P$.
(iii) Let $\langle f_n, \Omega_n \rangle \in P$ for a sequence of functions f_n with $f_n \to f$ normally on Ω, and an increasing sequence of domains $\Omega_1 \subset \Omega_2 \subset \ldots \Omega$ and $\Omega = \cup_{n \geq 1} \Omega_n$. Then the limit function $\langle f, \Omega \rangle \in P$ also.
(iv) If $\langle f, \mathbb{C} \rangle \in P$ then f is constant.

We will call such a property a Bloch property. Then for any domain Ω, any family of meromorphic functions $\mathcal{F}$ that satisfies P on Ω is normal on Ω.

Proof. We argue by contradiction. Assume $\mathcal{F}$ satisfies some property P as above on Ω, and $\mathcal{F}$ is not normal. By Lemma 6.1 and property P condition (ii), we can assume $\overline{\mathbb{D}} \subset \Omega$ and that $\mathcal{F}$ is not normal here.

Let $\rho_n > 0$, $z_n \in \overline{\mathbb{D}}$, $f_n \in \mathcal{F}$, $g_n(\zeta) = f_n(z_n + \rho_n \zeta)$, $R_n = (1 - |z_n|)/\rho_n$ and $g(\zeta)$ entire and (crucially) non-constant be as in Zalcman's lemma. By taking a subsequence, we can assume $R_n \to \infty$ is increasing, such that the sequence $D_n = \overline{B}(0, R_n)$ is increasing with limit $\mathbb{C}$.

By property (ii) with $\psi(\zeta) = z_n + \rho_n \zeta$, we have that the functions g_n satisfy P on D_n. Thus since $\langle g_n, D_n \rangle \in P$ for all n, we can apply condition (iii) to deduce that $\langle g, \mathbb{C} \rangle \in P$. But condition (iv) implies g must be constant, a contradiction. Thus $\mathcal{F}$ is normal as required. □

We see that we did not actually need condition (i) in our proof, as this is actually just a consequence of condition (iii). It is included only for convenience. We now see how this makes precise the example with Montel's theorem we saw at the beginning of the chapter.

Example 6.8. Let P be the property of meromorphic functions that satisfy

$$\langle f, \Omega \rangle \in P \text{ if and only if } |f(z)| \leq M(f, \Omega) \text{ for some constant } M(f, \Omega) > 0.$$

We see that conditions (i), (ii) and (iv) hold, however condition (iii) does not. For example, the functions $\{f_n(z) = 1/z + 1/n\}$ satisfy P on the annuli

$$A_n = \{z \in \mathbb{C} : 1/n \leq |z| < 1\}.$$

These functions are normally convergent to the meromorphic function $f(z) = 1/z$ on the punctured disc $\mathbb{D} \setminus \{0\}$ by Marty's theorem, as the spherical derivatives satisfy

$$f_n^{\#}(z) = \frac{1/|z|^2}{1 + |1/z + 1/n|^2} = \frac{1}{|z|^2 + |1 + z^2/n|^2} \leq 1 \quad \text{for all } n \in \mathbb{N}$$

The sequence of annuli A_n have limit $\mathbb{D} \setminus \{0\} = \cup A_n$. But the limit function $f(z) = 1/z$ is not bounded on $\mathbb{D} \setminus \{0\}$. It follows that this property does not satisfy condition (iii) and so P is not a Bloch property. This confirms what we expected, as we know that boundedness of each function is not sufficient for normality, considering Example 5.17.

Example 6.9. Fix $M > 0$ and let P be the property of meromorphic functions that satisfy

$$\langle f, \Omega \rangle \in P \textit{ if and only if } \sup_{z \in \Omega} |f(z)| \leq M.$$

P satisfies conditions (i), (ii) and (iv). If $|f_n(z)| \leq M$ for all $n \in \mathbb{N}$ and $z \in \Omega_n$ increasing with limit $\Omega = \cup \Omega_n$, then any limit function also satisfies $|f(z)| \leq M$ for all $z \in \Omega$, giving us condition (iii). Thus P is inducing a Bloch property and any family of holomorphic functions uniformly bounded on their domain is normal. This confirms what we already know by Montel's theorem.

While Zalcman's lemma gave us another equivalent condition for normality, Bloch's principle only tells us the sufficiency of certain conditions and does tell us anything about the other direction - as we see here uniform boundedness is sufficient for normality, but we know that only local boundedness is needed.

6.3 Montel's fundamental normality test

We now study a condition for normality, given by Paul Montel in 1912 [22]. Montel's original proof made use of the modular lambda function, a highly symmetric holomorphic function on the upper half-plane. Zalcman's lemma provides a comparatively elementary proof. This approach comes from [5, 11, 39].

Definition 6.10. We say a family $\mathcal{F} \subset \mathcal{M}(\Omega)$ *omits* a value $a \in \mathbb{C}_\infty$ on Ω if it is not in the image of any $f \in \mathcal{F}$ on Ω. Equivalently, the equation $f(z) = a$ has no solutions for any $f \in \mathcal{F}$ and $z \in \Omega$.

Theorem 6.11 (Fundemental Normality Test (FNT))**.** *Let $\mathcal{F} \subset \mathcal{M}(\Omega)$ be a family of meromorphic [holomorphic] functions which omits three [two] distinct values on Ω. Then $\mathcal{F}$ is normal.*

Proof. We only prove the meromorphic case, since the holomorphic case follows from the meromorphic case immediately by viewing holomorphic functions as meromorphic functions which omit ∞ everywhere.

By Corollary 6.3 we can assume without loss of generality that the three values omitted are $0, 1, \infty$. We proceed by contradiction, assuming $\mathcal{F}$ is not normal. Then $\mathcal{F}$ is not normal in some region of Ω so by Lemma 6.1 we can further assume without loss of generality that $\Omega = \mathbb{D}$. All $f \in \mathcal{F}$ are holomorphic and non-zero on $\mathbb{D}$, so we can define a holomorphic 2^nth root of each f. Let the family $\mathcal{F}_n$ be the family of all 2^nth roots of functions in $\mathcal{F}$, such that $g \in \mathcal{F}_n$ if and only if $g^{2^n} = f$ for some $f \in \mathcal{F}$. Since $\mathcal{F}$ omits 1, $\mathcal{F}_n$ omits all 2^nth roots of unity $e^{2k\pi i/2^n}$. Observe that for $g \in \mathcal{F}_n$

$$\begin{aligned} g^{\#} &= \frac{|(f^{1/2^n})'|}{1 + |f^{1/2^n}|^2} \\ &= \frac{1}{2^n} \frac{|f^{(1/2^n-1)}||f'|}{1 + |f|^{2/2^n}} \\ &= \frac{1}{2^n} \frac{|f^{(1/2^n-1)}|}{1 + |f|^{2/2^n}} (1 + |f|^2) f^{\#} \\ &= \frac{1}{2^n} \frac{|f|^{-1}(1 + |f|^2)}{|f|^{-1/2^n} + |f|^{1/2^n}} f^{\#} \\ &= \frac{1}{2^n} \frac{|f|^{-1} + |f|}{|f|^{-1/2^n} + |f|^{1/2^n}} f^{\#} \\ &\geq \frac{1}{2^n} f^{\#}. \end{aligned}$$

Here we make use of the inequality $a^{-1} + a \geq a^x + a^{-x}$ for $a > 0$ and $0 < x < 1$ - see Appendix. Thus by Marty's theorem, we have $\mathcal{F}$ not normal implies $\mathcal{F}_n$ is not normal for any fixed $n \in \mathbb{N}$.

Let G_n be the non-constant entire function attained by Zalcman's lemma applied to $\mathcal{F}_n$. From the lemma, we know that these satisfy $G_n^{\#}(z) \leq 1$ and thus $\{G_n\}$ form a normal family by Marty's theorem. By taking a suitable subsequence, let $G_n \to G$ normally be the (once again crucially) non-constant entire limit function of the sequence. G is entire by Theorem 5.8.

By Hurwitz's theorem, we know that each G_n omits the 2^nth roots of unity, as they are normally convergent limit functions from the family $\mathcal{F}_n$ which omitted these values. Now by Hurwitz's theorem again, G omits all 2^nth roots of unity $e^{2\pi ki/2^n}$ for all $n, k \in \mathbb{N}$.[3] But these are dense on the unit circle and since G is holomorphic we must have $G(\mathbb{C})$ open so either $|G(z)| < 1$ or $|G(z)| > 1$. Since 0 is an omitted value also, then $|1/G(z)| < 1$ is entire non-constant also. In either case, Liouville implies G must be constant, a contradiction. Thus $\mathcal{F}$ must be normal as required. □

The above proof is a showcase of the power of Hurwitz's theorem and Zalcman's lemma. Starting with only 3 omitted values we apply Zalcman's lemma once to attain entire functions G_n which omit 2^nth roots of unity then apply Zalcman's lemma a second time to attain an entire function which omits all of the unit circle. Observe that we really did need at least 3 values omitted, all of $0, 1$ and ∞ were crucial. We needed both 0 and ∞ omitted to allow a holomorphic root to exist and needed 1 omitted to make $\mathcal{F}_n$ omit the 2^n roots of unity which lead to the unit circle being omitted in our final limit function G. The omission of 0 was once needed again at the end to deduce that $1/G$ was entire.

[3] Even though we passed to a subsequence, since we have $n \to \infty$ and the 2^nth roots of unity include all previous roots of unity 2^m for $m \leq n$, we will still omit them all.

Montel's result gives a very powerful sufficient condition for normality, however, it is not at all necessary. For example, the family containing merely the single holomorphic function $\{f(z) = z\}$ omits no values in $\mathbb{C}$ (or only one value ∞ in $\mathbb{C}_\infty$ viewed as a meromorphic function) but is normal. We demonstrate Montel's FNT with the following examples.

Example 6.12. Let $\mathcal{F} \subset \mathcal{H}(\mathbb{D})$ be the family of functions such that

$$\mathcal{F} := \{f \in \mathcal{H}(\mathbb{D}) \mid \Re(f(z)) \neq \Im(f(z))^3 \text{ for all } z \in \mathbb{D}\}.$$

Since $\mathcal{F}$ omits all values $z = x + iy$ for all $x = y^3$, then by the FNT, $\mathcal{F}$ is normal. We verify this by making use of the Riemann mapping theorem. Observe that by continuity, we must have that the image of each function is entirely contained in only one of Ω_1 or Ω_2, as the line $x = y^3$ partitions the plane.

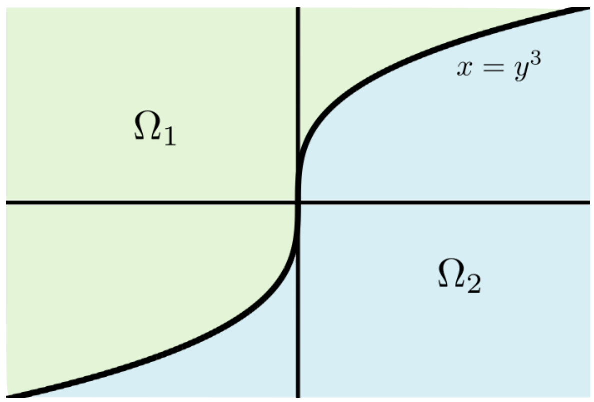

Figure 6.1: The image $f(\mathbb{D}) \subset \Omega_1 \cup \Omega_2$ for all functions in $f \in \mathcal{F}$.

It follows that we can write $\mathcal{F}$ as a disjoint union

$$\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2 \quad \mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{H}(\mathbb{D}),$$

$$\mathcal{F}_1 = \{f \in \mathcal{H}(\mathbb{D}) \mid f(\mathbb{D}) \subset \Omega_1\} \quad \mathcal{F}_2 = \{f \in \mathcal{H}(\mathbb{D}) \mid f(\mathbb{D}) \subset \Omega_2\}.$$

It is clear that Ω_1 and Ω_2 are simply connected, so by the Riemann mapping theorem, there exist biholomorphisms $h_1 : \Omega_1 \to \mathbb{D}$ and $h_2 : \Omega_2 \to \mathbb{D}$. Now, as in Example 6.4, we can apply Lemma 6.2 to deduce that the families $h_1 \circ \mathcal{F}_1$ and $h_2 \circ \mathcal{F}_2$ are normal if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are respectively. By Montel's theorem, each of these is normal. It is clear that a finite union of normal families is normal, as any sequence must have infinitely many terms from one of the families, so we can extract a subsequence from there. It follows that $\mathcal{F}$ is normal, confirming the result the FNT gives us.

Example 6.13. It was crucial to our proof that all values omitted by each function were the same, and asking that each function only omit three values is certainly not sufficient for normality. This is seen once again with the family $\mathcal{F} \subset \mathcal{M}(\mathbb{D})$ from Example 5.17 defined by

$$\mathcal{F} = \{nz : n \in \mathbb{N}\}.$$

We have shown that this family is not normal. Each function omits an infinitude of points given by $\mathbb{C}_\infty \setminus B(0, n)$, but every value in $\mathbb{C}$ is attained by all functions with suitably large n. Only the point ∞ is omitted by every function on $\mathbb{D}$.

7 The Picard Theorems

In our final chapter, we make use of the FNT to prove some of the most beautiful results in complex analysis, the Picard theorems. We follow proof of these results from [7, 11, 40].

7.1 The big Picard theorem

We first prove the big Picard theorem. This is a considerable strengthening of the Casorati-Weierstrass theorem, which states that the image of any holomorphic function in the neighbourhood of an essential singularity is dense in $\mathbb{C}$. A dense subset can still omit almost all points, the big Picard theorem strengthens this to tell us we can omit only a single value at most.

Theorem 7.1 (The Big Picard Theorem)**.** *Suppose a function $f(z)$ is holomorphic on a punctured neighbourhood $\{z : 0 < |z - z_0| < R\}$, with an essential singularity at z_0. Then on any neighbourhood of z_0, f assumes every value in $\mathbb{C}$ here, with one possible exception, infinitely many times.*

Proof. By considering $f(z - z_0)$ we may assume $z_0 = 0$. We consider the punctured neighbourhood $D := \{z : 0 < |z| < R\}$. We show that if at least two values are omitted by f on D, then f must in fact have a removable singularity or pole at z_0, rather than an essential singularity. Let $a, b \in \mathbb{C}$ be two distinct values omitted by f on D. Consider the sequence of holomorphic functions on D

$$g_n(z) = f(z/n).$$

Since the holomorphic family $\{g_n\}$ omits the values a, b then by the FNT it is a normal family. By passing to a suitable subsequence, let $g_n \to g$ normally be the limit function. By Theorem 5.8 either g is holomorphic on D or $g \equiv \infty$.

First, assume g is holomorphic on D, so we also have Euclidean compact convergence here. Then g is bounded on any compact subset of D, including the circle $|z| = R/2$. Let $M > 0$ be such that $|g(z)| < M$ for all z with $|z| = R/2$. Since the g_n converge to g uniformly on compact subsets of D, for some $N \in \mathbb{N}$ we have that for all $n \geq N$,

$$|f(z/n)| = |g_n(z)| < M + 1 \text{ for } |z| = R/2$$
$$\implies |f(z)| < M + 1 \text{ for } |z| = R/2n.$$

Since this is a bound on all such concentric circles as $n \to \infty$ and $R/2n \to 0$, then, by the max modulus principle, f is bounded on the annuli $\{z : R/2n < |z|R/2\}$. But these annuli have limit $\{z : 0 < |z| < R/2\}$, and so $|f(z)| \leq M + 1$ here. Then by the Riemann extension theorem, $z = 0$ is a removable singularity, and f extends to be holomorphic here.

Alternatively, assume $g_n \to g \equiv \infty$ on D. By Proposition 5.7, $1/g_n \to 1/g \equiv 0$ normally. The same argument as above tells us that $1/f$ is holomorphic and bounded on a punctured neighbourhood of $z = 0$, so by the Riemann extension theorem $1/f$ extends to be holomorphic here with value $1/f(0) = 0$. Thus f extends to be meromorphic with a pole at $z = 0$ since $\lim_{z \to 0} f(z) = \infty$.

In either case, the omitted values contradict that $z = 0$ is an essential singularity. It follows that in any punctured neighbourhood of an essential singularity, there can be at most one omitted value. For the "infinitely often", we note that if two values occurred only finitely often then choosing a punctured disc with a suitably small radius would cause these to be omitted and the above argument would imply z_0 was not an essential singularity. Thus every value must occur infinitely often in any neighbourhood, with at most one exception. $\square$

The brilliance of this result is evident, that a holomorphic function attains all of $\mathbb{C}$ but one infinitely often in an arbitrarily small punctured neighbourhood seems impossible, and yet it occurs around any isolated essential singularity.

Example 7.2. To get a grasp of how this can happen we view a domain colouring of a function with an isolated essential singularity. Consider the function

$$f(z) = e^{1/z}; \quad z \in \mathbb{C} \setminus \{0\}.$$

Since f is well defined for $z \neq 0$ then there is an isolated singularity here. Taking $z \to 0^+$ along the positive real axis gives limit ∞, while approaching along the negative real axis $z \to 0^-$ gives $f(z) \to 0$. Thus since f is not bounded in a neighbourhood of 0 and does not uniformly approach infinity here, $z = 0$ cannot be a removable singularity or pole and thus must be an essential singularity of f. Observe that for no value of $z \in \mathbb{C} \setminus \{0\}$ does $e^{1/z} = 0$, so this value is omitted. By the big Picard theorem, on any neighbourhood of $z = 0$, f attains every value in $\mathbb{C}$ apart from 0 infinitely often.

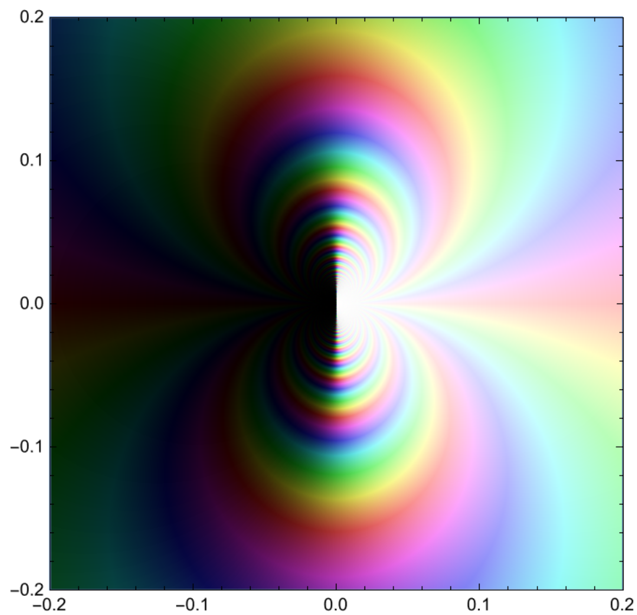

Figure 7.1: HSL domain colouring of $e^{1/z}$ around $z = 0$.

We can see the structure of this infinite assuming of each point in Figure 7.1. Here we use HSL domain colouring such that for each z in the domain we colour the hue by a factor of $\arg(f(z))$, saturation is held constant, and the lightness is determined by the modulus $|f(z)|$ such that points go whiter as the modulus approaches infinity and black towards zero. Approaching from the left has points of modulus closer to 0 and the right has points with modulus approaching infinity. As we approach along the positive or negative imaginary axis we are varying through $e^{1/it} = e^{-i/t}$ as $t \to 0^+$ or 0^-, so we cycle through all possible arguments infinitely often. Combining all possible arguments and all possible moduli > 0, we can attain any complex number apart from 0.

Example 7.3. It is not necessary that the function should omit at least one value, as we see with the function

$$f(z) = \sin(1/z); \quad z \in \mathbb{C} \setminus \{0\}.$$

Since this is well defined for $z \neq 0$, f has an isolated singularity here. Approaching along $z_n = 1/n\pi \to 0$ gives $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$, but the sequence $z_m = 1/(m\pi + \pi/2) \to 0$ does not give the same value, since $\sin(m\pi + \pi/2) = (-1)^m \not\to 0$. This means the limit is undefined as we approach $z = 0$ so there cannot be a removable singularity or pole here, as in either case, we would have convergence to some value or infinity. Again we must have an essential singularity. By the big Picard theorem, on any punctured neighbourhood of $z = 0$, $\sin(1/z)$ assumes every value in $\mathbb{C}$ apart from one possible exception, infinitely often.

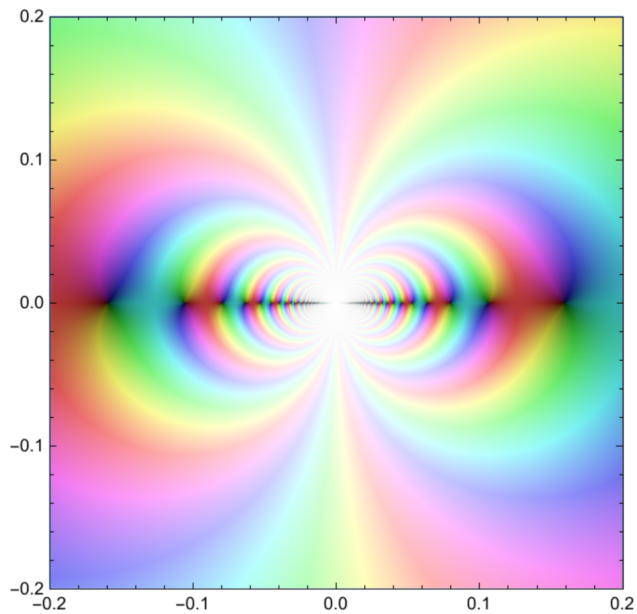

Figure 7.2: HSL domain colouring of $\sin(1/z)$ around $z = 0$.

To show that $\sin(1/z)$ assumes *every* value in $\mathbb{C}$ infinitely often on any punctured neighbourhood, let $w \in \mathbb{C}$ be arbitrary. We want to solve

$$\sin\left(\frac{1}{z}\right) = \frac{e^{i/z} - e^{-i/z}}{2i} = w.$$

Since $e^{i/z} \neq 0$ for all $z \in \mathbb{C}$ we can multiply by this to give us a quadratic in $e^{i/z}$

$$\left(e^{i/z}\right)^2 - 2iw\left(e^{i/z}\right) + 1 = 0.$$

By the fundamental theorem of algebra, this has a solution $e^{i/z} = a \in \mathbb{C}$. By construction, it is clear that regardless of the value of w we have $a \neq 0$, as this would not solve the above polynomial. By the previous example, we can find infinitely many solutions z to the equation $e^{i/z} = a$, that are arbitrarily close to zero. These z solve $\sin(1/z) = w$, and thus $\sin 1/z$ attains *every* value in $\mathbb{C}$ infinitely often on with no exceptions on any punctured neighbourhood of 0. This can also be shown without the previous example by using the periodicity of sin to note that if there is a single solution, we can easily find z with an arbitrarily small modulus which solves this.

7.2 The little Picard theorem

The little brother to this theorem - which was actually proved 5 months prior - is now proven as a consequence of the big Picard theorem. We first need a short lemma about entire functions.

Lemma 7.4. *If f is entire and $\lim_{z\to\infty} f(z) = \infty$, then f is a polynomial.*

Proof. Since f is entire it can be represented as a Taylor series which converges on all $\mathbb{C}$, with the limit of the Taylor series also approaching ∞ as $z \to \infty$ from the statement.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

We want to show this sum has finitely many non-zero coefficients. Define $g(z)$ by

$$g(z) = f(1/z) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$$

This Taylor series must converge for all $z \in \mathbb{C} \setminus \{0\}$. Since $\lim_{z\to 0} g(z) = \infty$ then $z = 0$ is a pole of g, so there are only finitely many negative power terms of z in its Laurent series here. Thus there exist finitely many $a_n \neq 0$, so f is polynomial as required. □

Theorem 7.5 (The Little Picard Theorem)**.** *If f is an entire function [meromorphic function on $\mathbb{C}$] which omits two [three] values in $\mathbb{C}$ [in $\mathbb{C}_\infty$], then f must be constant.*

Proof. If $f(z)$ is a meromorphic function on $\mathbb{C}$ that omits some value $a \in \mathbb{C}$, then the function $h = 1/(f - a)$ is entire and omits one less value than f in $\mathbb{C}$. This is because we have composed f with a Möbius transform taking a to infinity, and these are bijections of $\mathbb{C}_\infty$ so the total number of values omitted in $\mathbb{C}_\infty$ must be preserved under composition. Since h is constant if and only if f is constant, it suffices to prove the entire case only.

As in the lemma, consider $g(z) = f(1/z)$ and consider the point $z = 0$ as an isolated singularity of g. Since f omits two values on $\mathbb{C}$ then g omits two values on any punctured neighbourhood of 0, so $z = 0$ must be a pole or removable singularity of g by big Picard.

If g has a pole at $z = 0$ then $\lim_{z\to\infty} f(z) = \lim_{z\to 0} g(z) = \infty$ so by Lemma 7.4 f is polynomial. But by the fundamental theorem of algebra, any non-constant polynomial attains every value in $\mathbb{C}$ since for each $a \in \mathbb{C}$, $f(z) - a$ must have as many roots counted with multiplicity as the degree. Thus the only way for f to omit two values is to be a constant polynomial.

If g has a removable singularity at $z = 0$ then g extends to be entire. Thus g is bounded on the unit disc $\mathbb{D}$ by $|g(z)| \leq M$ for some $M > 0$. Thus $|f(z)| = |g(1/z)| \leq M$ for $|z| > 1$ and there is clearly also a bound $|f(z)| < K$ for $|z| \leq 1$, since it is holomorphic here. This implies f is bounded and entire, so constant by Liouville.

In either case, we have f is necessarily constant as required. □

The order the results in these last two chapters have been presented is reversed compared to how they developed in history, as the more modern work by Zalcman enables us to give elementary proofs of earlier results. Picard first announced his "little" theorem on May 19th 1879. His second announcement of his more powerful version of the theorem followed only five months later on October 20 1879 [40]. Picard's original proof in [24] made use of the modular lambda function mentioned in the previous chapter. In 1907 Montel published his first paper on the topic of normal families, detailing his theorem we saw in Chapter 3. His FNT was published in 1912 with his proof making use of the modular lambda function also.

Observe that omitting three values forces a meromorphic function on $\mathbb{C}$ to be constant and that the FNT tells us all functions omitting the same three values causes a meromorphic family to be normal. This suggests that the little Picard theorem is inducing a Bloch property for the FNT, as we touched on at the end of the previous chapter. Although we have proved Picard by using the FNT, we can verify that this indeed is the case. Fix distinct $\alpha, \beta, \gamma \in \mathbb{C}_\infty$ and let P be the property

$$\langle f, \Omega \rangle \in P \textit{ if and only if } f \textit{ omits } \alpha, \beta, \gamma \textit{ on } \Omega.$$

For Bloch's principle condition (i) we note that if a function omits a value on Ω then it clearly omits this value on any $D \subset \Omega$.

For Bloch's principle condition (ii), if f omits three distinct values $\alpha, \beta, \gamma \in \mathbb{C}_\infty$ on Ω and $\psi(z) = az + b$, then the function $f \circ \phi$ also omits the α, β, γ on $\psi^{-1}(\Omega)$, since the values passed into the f part of $f \circ \psi$ are merely elements of Ω as the translation of the domain is undone.

Condition (iii) is a little more complex. Assume $f_n \to f$ normally on Ω and Ω_n is an increasing sequence of subsets of $\mathbb{C}$ with limit $\Omega = \cup \Omega_n$. Assume $\langle f_n, \Omega_n \rangle \in P$ for all $n \in \mathbb{N}$, then fixing the index Ω_n we observe $\langle f_m, \Omega_n \rangle \in P$ for all $m \geq n$ by property (i) and the fact that $\Omega_n \subset \Omega_m$ for all $m \geq n$. By Hurwitz's theorem[1] if f_m omits a value $\alpha \in \mathbb{C}_\infty$ on Ω_n for all $m \geq n$, then the normally convergent limit function f also omits this value on Ω_n. Thus we deduce $\langle f, \Omega_n \rangle \in P$ for all $n \in \mathbb{N}$. Now assume $\langle f, \Omega \rangle \notin P$, thus without loss of generality there exists $z_0 \in \Omega$ such that $f(z_0) = \alpha$. But then there exists $m \in \mathbb{N}$ such that $z_0 \in \Omega_m$, since $\Omega = \cup \Omega_n$. As f attains the value α on Ω_m, we have $\langle f, \Omega_i \rangle \notin P$, contradicting our above deduction. It follows that we must have $\langle f, \Omega \rangle \in P$.

Finally, condition (iv) follows from Picard's little theorem, making P a Bloch property and confirming Montel's FNT result, as α, β, γ were arbitrary. It was from results such as Montel's FNT and Picard's theorem, along with Montel's theorem from Chapter 3 paired with Liouville's theorem, that lead to the formulation of Bloch's heuristic principle which was later made rigorous by Zalcman.

[1] Note that we can only apply Hurwitz if all three values omitted are the same for every function, so we could not have P = "f omits some collection of 3 values", we really do need that they are the same.

8 Conclusions and Further Reading

This report has aimed to give an introduction and overview of the theory regarding normal families in complex analysis. We began by proving the Arzelà–Ascoli theorem, giving us a condition for the normality of a family in the most general case, that of continuous functions. It cannot be overstated how crucial the Arzelà–Ascoli theorem is, with every single further normality criterion proved here having some dependency on it. It directly assisted us in proof of Montel's theorem and Marty's theorem, and by extension was necessary to all of the results presented in Chapter 6.

Equicontinuity was introduced as part of the Arzelà–Ascoli theorem for families of continuous functions. We then proved that only local boundedness was needed for families of holomorphic functions. Of course, any normal family of holomorphic functions is equicontinuous as we know this to be a necessary condition. Our proof of Montel's theorem involved proving that equicontinuity followed from local boundedness combined with the holomorphic structure of each function. Similarly, any meromorphic normal family is spherically equicontinuous, although the equivalence condition provided by Marty does not make this explicit in its statement. Arzelà–Ascoli gave us an equivalent set of conditions for normality, so these hold in any case regardless of which criterion we have applied to deduce normality.

In two instances, we have seen the utility of knowing a certain family is normal, through the Riemann mapping theorem and the big Picard theorem. The Riemann mapping theorem itself has many applications, in areas such as fluid mechanics. By taking a problem in fluid dynamics defined on some challenging domain and transforming the domain and equations by the same biholomorphic map, a solution can sometimes be found and then translated back. For further information on conformal mapping in fluid mechanics, see [30].

Proved in 1975, Zalcman's lemma "built a bridge" between normal families and another area of complex analysis, Nevanlinna theory. Nevanlinna theory is the study of the asymptotic behaviour of the number of solutions to the equation $f(z) = a$ for meromorphic f, as a varies. Zalcman's lemma has been used to prove results in Nevanlinna theory using normal family methods, as well as derive further normality criteria using the reverse direction. Nevanlinna theory itself has applications including the study of minimal surfaces. These are surfaces that have locally minimal area, such that every point on the surface has a neighbourhood which has minimal area compared with all other surfaces of the same boundary. Further information regarding this connection to Nevanlinna theory can be found in [29, 31].

One of the first applications of Montel's FNT was in 1925, in the field of holomorphic dynamics. This is the study of the behaviour of points in $\mathbb{C}$ under repeated application of a certain function. In many cases this behaviour can be extremely chaotic in nature, leading to complicated fractal sets when asking which points satisfy a certain condition under iteration. By considering the family formed by all repeated iterations of a certain function, normality can be used to deduce information about the Fatou and Julia sets of a function f. These are the sets of points in $\mathbb{C}_\infty$ which are "well-behaved" and "chaotic" under iteration of f respectively. Further information regarding normal families in holomorphic dynamics can be found in [11].

At the end of Chapter 7 we confirmed the result of the FNT by showing that the little Picard theorem was inducing a Bloch property. In fact, we can use little Picard to *extend* the FNT to a more general form below, by reformulating the property it induces. This allows each function to omit different values, so long as they all satisfy a certain condition. Proof of this can be found in [29].

Theorem 8.1 (Extended FNT)**.** *Let $\mathcal{F} \subset \mathcal{M}(\Omega)$ be a family of meromorphic functions such that each function $f \in \mathcal{F}$ omits three distinct values $\alpha_f, \beta_f, \gamma_f \in \mathbb{C}_\infty$ which satisfy*

$$\chi(\alpha_f, \beta_f) \cdot \chi(\beta_f, \gamma_f) \cdot \chi(\alpha_f, \gamma_f) \geq \epsilon$$

for some $\epsilon > 0$ independent of f. Then $\mathcal{F}$ is normal.

We have proven some of the most well-known criteria for normality in this report, however, many more have been proven to date. Bloch's principle can be used to derive other normality conditions, further details regarding many of these can be found in [5].

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Appendix

Inequality in proof of FNT, with help from [18].

We want to show $a^{-1} + a \geq a^x + a^{-x}$ for $a > 0$ and $0 < x < 1$.

Consider

$$f(x) = a^x + a^{-x}.$$

For fixed a, the derivative is given by

$$f'(x) = \log(a)a^x + \log(a)a^{-x} = \log(a)(a^x + a^{-x})$$

For $a \geq 1$ we know for $x > 0$ that

$$\log(a) \geq 0 \text{ and } (a^x + a^{-x}) \geq 0 \implies f'(x) \geq 0.$$

Similarly for $0 < a \leq 1$ and $x > 0$, we know

$$\log(a) \leq 0 \text{ and } (a^x + a^{-x}) \leq 0 \implies f'(x) \geq 0.$$

Thus $f'(x) \geq 0$ for each fixed $a > 0$ for all $x \in (0, \infty)$. It follows that the function

$$g(x) = a + a^{-1} - a^x - a^{-x}$$

is decreasing for each fixed $a > 0$, for all $x \in (0, \infty)$. Since we have $g(1) = 0$, it follows that $g(x) \geq 0$ for each $a > 0$ and $0 < x < 1$, giving the required inequality.

Mathematica Code

Figure 1.1 - Bolzano Weierstrass cube

```
split[cube_]:=Module[{min, max}, min = cube[[1]];
max = cube[[2]];
x = (max[[1]] − min[[1]])/2;
y = (max[[2]] − min[[2]])/2;
z = (max[[3]] − min[[3]])/2;
{Cuboid[min, {x, y, z}],
Cuboid[{x, 0, 0}, {2x, y, z}],
Cuboid[min +{0, y, 0}, min +{x, 2y, z}],
Cuboid[min +{x, y, 0}, min +{2x, 2y, z}],
Cuboid[min +{0, 0, z}, min +{x, y, 2z}],
Cuboid[min +{x, 0, z}, min +{2x, y, 2z}],
Cuboid[min +{0, y, z}, min +{x, 2y, 2z}],
Cuboid[min +{x, y, z}, max]}];
pointsA = RandomVariate[UniformDistribution[], {10000, 3}];
mean = {7/16, 7/16, 5/16};
cov = 0.001 ∗ IdentityMatrix[3];
points = RandomVariate[MultinormalDistribution[mean, cov], 5000];
pointsB = Select[points, And@@Thread[0<=#<=1]&];
cube = Cuboid[{0, 0, 0}, {1, 1, 1}];
subcube1 = split[cube][[1]];
subcube2 = split[subcube1][[8]];
subcube3 = split[subcube2][[4]];
Graphics3D[{EdgeForm[Thin], Opacity[0.3], cube,
Opacity[0.1], Red, subcube1,
Opacity[0.1], Blue, subcube2,
Opacity[0.1], Blue, subcube3,
Opacity[0.5], PointSize[Tiny], Black, Point[pointsB],
```

```
Opacity[0.5], PointSize[Tiny], Black, Point[pointsA]}]
```

Figure 5.1 - stereographic projection

```
Graphics3D[{Opacity[0.6], LightBlue, Sphere[],
Opacity[0.4], LightBlue,  Cuboid[{−3, −3, 0}, {3, 3, 0}],
Opacity[0.5], Black, Arrow[{{0, 0, 0}, {0, 0, 2}}],
Opacity[0.5], Arrow[{{0, 0, 0}, {0, 0, −2}}],
Opacity[0.5], Arrow[{{0, 0, 0}, {0, 3, 0}}],
Opacity[0.5], Arrow[{{0, 0, 0}, {0, −3, 0}}],
Opacity[0.5], Arrow[{{0, 0, 0}, {3, 0, 0}}],
Opacity[0.5], Arrow[{{0, 0, 0}, {−3, 0, 0}}],
Opacity[1], PointSize[Medium], Point[{0, 0, 1}],
Text[Style["N=(0, 0, 1)", FontSize->11], {0, 0, 1.2}],
Opacity[1], Point[{1.5, −2, 0}],
Text[Style["z=(x, y, 0)", FontSize->11], {1.5, −2, −0.2}],
Opacity[1], Point[{0.4138, −0.5517, 0.7241}],
Text[Style["P(z)", FontSize->11], {0.4138 + 0.1, −0.5517 − 0.1, 0.7241 + 0.2}],
Opacity[1], Line[{{0, 0, 1}, {2.25, −3, −0.5}}]}]
```

Figure 5.2 - proof of Marty's theorem

```
p1 = {1, 6};
p2 = {3, 2};
sp1 = {2 ∗ p1[[1]]/(1 + p1[[1]]^2 + p1[[2]]^2), 2 ∗ p1[[2]]/(1 + p1[[1]]^2 + p1[[2]]^2),
(p1[[1]]^2 + p1[[2]]^2 − 1)/(p1[[1]]^2 + p1[[2]]^2 + 1)};
sp2 = {2 ∗ p2[[1]]/(1 + p2[[1]]^2 + p2[[2]]^2), 2 ∗ p2[[2]]/(1 + p2[[1]]^2 + p2[[2]]^2),
(p2[[1]]^2 + p2[[2]]^2 − 1)/(p2[[1]]^2 + p2[[2]]^2 + 1)};
Hemisphere = ParametricPlot3D[{Cos[v], Cos[u]Sin[v], Sin[u]Sin[v]}, {u, 0, Pi}, {v, 0, π},
PlotStyle->{LightBlue, Opacity[0.2]},  MeshStyle->{Gray, Opacity[0.3]}];
SphPoints = Graphics3D[{Opacity[1], Black, PointSize[0.015], Point[sp1],
```

```
Text [Style ["f(z0)", FontSize->16] , sp1 + {−0.07, −0.07, 0.1}] ,
Point[sp2],
Text [Style ["f(z1)", FontSize->16] , sp2 + {0.15, −0.15, 0.1}] ,
Text[Style["f(γ)", FontSize->16, FontColor->Purple], (sp2 + sp1)/2 + {0.15, 0.2, 0.1}],
Thickness[0.004], Line[{sp1, sp2}]}];
n = 8;
A = 0.3;
SphPath = ParametricPlot3D[{2 ∗ (1 + 2 ∗ t + ASin[nPit])/(1 + (1 + 2 ∗ t + ASin[nPit])^2+
(6 − 4 ∗ t + ASin[nPit])^2),
2 ∗ (6 − 4 ∗ t + ASin[nPit])/(1 + (6 − 4 ∗ t + ASin[nPit])^2 + (1 + 2 ∗ t + ASin[nPit])^2),
((6 − 4 ∗ t + ASin[nPit])^2 + (1 + 2 ∗ t + ASin[nPit])^2 − 1)/
((1 + (6 − 4 ∗ t + ASin[nPit])^2 + (1 + 2 ∗ t + ASin[nPit])^2))}, {t, 0, 1}, PlotStyle->Purple];
Show[SphPoints, SphPath, Hemisphere, Boxed->False]
```

Figures 7.1 and 7.2 - HSL domain colourings

Below is the code to generate Figure 7.1. For Figure 7.2, one must simply exchange "Exp" with "Sin" on the first line. Some of this code including the specific colour function to express HSL domain colouring in terms of the HSV format of the Hue function is adapted from [28].

```
f[z_]:=Exp[1/z];
resolution = 900;
colourHSL =
ParametricPlot[{Re[x + Iy], Im[x + Iy]}, {x, −0.2, 0.2}, {y, −0.2, 0.2},
ColorFunction → Function[{x, y}, Hue[Arg[f[(x + Iy)]]/(2Pi),
1, 1 − 2/Pi ArcTan[Log[1 + 1/Abs[f[x + Iy]]]],
1 − 2/Pi ArcTan[Log[1 + Abs[f[x + Iy]]]/2]]],
ColorFunctionScaling → False,
PlotRange → {{−0.20, 0.20}, {−0.20, 0.20}},
PlotPoints → resolution,
FrameStyle → Black,  Axes → False]
```